

Pricing of Call Options on Foreign Exchange

by

Alireza Javaheri

Submitted to the Department of Electrical Engineering and
Computer Science

in partial fulfillment of the requirements for the degree of

Master of Science in Electrical Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1994

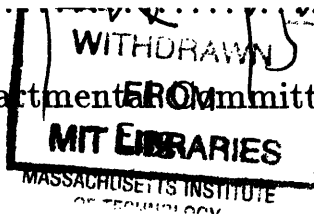
© Alireza Javaheri, MCMXCIV. All rights reserved.

The author hereby grants to MIT permission to reproduce and
distribute publicly paper and electronic copies of this thesis
document in whole or in part, and to grant others the right to do so.

Author
Department of Electrical Engineering and Computer Science
May 6, 1994

Certified by
Jeremy F. Shapiro
Professor of Operations Research and Management
Thesis Supervisor

Accepted by
Frederic R. Morgenthaler
Chairman, Departmental Committee on Graduate Students



JUL 13 1994

Pricing of Call Options on Foreign Exchange

by

Alireza Javaheri

Submitted to the Department of Electrical Engineering and Computer Science
on May 6, 1994, in partial fulfillment of the
requirements for the degree of
Master of Science in Electrical Engineering

Abstract

The standard Black and Scholes formula for currency options assumes constant interest rates. This thesis demonstrates how to generalize the approach to incorporate varying interest rates, using a stochastic method suggested by Grabbe.

In particular, we discretize the equations by using the explicit finite difference method, in order to obtain a decision tree similar to the well known binary lattice.

We then implement this tree on computer and obtain a program determining different call option prices, given the boundary conditions. The results are then saved in files and plotted, so that one will be able to verify the coherence of the model and compare these results with the constant interest rate case.

Finally we reverse the problem and deduce the value of a parameter (specially the exchange rate volatility) from the call price real value.

Thesis Supervisor: Jeremy F. Shapiro

Title: Professor of Operations Research and Management

Acknowledgments

I would like to thank my thesis supervisor Prof. Shapiro who gave me the possibility of applying my technical skills to a practical financial problem and discover the art of implementation of abstract equations on computers.

Contents

1 Theory page 5

1.1 Introduction page 5

1.2 A Black and Scholes formula with varying interest rates page 8

1.3 Application of the Explicit Finite Difference Method page 13

1.4 Conclusion page 26

2 Implementation page 27

2.1 The Parameters page 27

2.2 The Decision Tree page 31

2.3 Boundary Values page 35

2.4 Computational Experience page 36

2.5 Other Parameters page 44

2.6 Sequential Decision System page 50

2.7 An Exact formula page 52

2.8 Implied Volatility page 56

3 Conclusion page 57

References

1 Theory

1.1 Introduction

Let us consider a multi-national corporation having revenues in the future in a foreign currency at a specific date. The company would like to have the possibility of choosing between Dollars and the foreign currency, depending on the value of the exchange rate. This problem could be treated as a sequential decision problem. Klaassen-Shapiro and Spitz [1990] have presented a stochastic programming method where the probabilistic tree contains different states in which the value of the exchange rate between the dollar and the foreign currency is chosen. They choose dollar revenue targets for each terminal node and determine the option price so as to minimize the expected discounted cost of meeting these revenue targets.

The options considered on the foreign exchange are *European*, which means that they could not be exercised before the expiration day. The theory to be used is the one developed by Black and Scholes [1973] for options on stocks, which we shall call *standard* options in opposition to options on foreign currency.

In their paper Biger and Hull [1983], give the price of an European call option c , at a given time t between the present time and the expiration date T , by the standard Black and Scholes formula, namely;

$$c = Se^{-FCiT} \mathcal{N}\left(\frac{\ln(S/E) + [(USi - FCi + (V^2/2)]T}{V\sqrt{T}}\right) - Ee^{-USiT} \mathcal{N}\left(\frac{\ln(S/E) + [(USi - FCi - (V^2/2)]T}{V\sqrt{T}}\right) \quad (1)$$

where S represents the domestic currency price of a unit of foreign exchange at time t , E is the currency exercise price of an option on foreign currency, \mathcal{N} the cumulative normal distribution function with zero mean and unit variance, USi, FCi the domestic and foreign interest rates, and V the volatility of the exchange rate's Brownian Motion.

Then they develop their sequential decision model for selecting the right currency options. For each possible value of the exchange rate the model selects an options strategy maximizing the level of dollar revenues resulting

from the exchange of a given *fixed* amount of foreign currency.

Finally, they construct their stochastic system describing the evolution of the exchange rate over which the decision model is developed.

The exercise price E is of course fixed and chosen by the decision maker, and so are the expiration time T and the exchange rate volatility V . But the issue is the choice of the domestic and foreign interest rates USi and FCi . In the paper by Klaassen-Shapiro-Spitz, the domestic interest rate USi is chosen to be fixed and the foreign interest rate is evaluated by this quantity and the exchange rate, based on the assumption of no arbitrage which implies:

$$S_{ts}(1 + FCi_{ts}) = (1 + USi_{ts})\mathcal{E}[S_{t+1,v}] \quad (2)$$

where t represents the stage and s a possible scenario in this stage, and \mathcal{E} is the expectation upon all possible scenarios. This equation has a financial interpretation: a unit of domestic currency that is exchanged immediately and then invested for one period yields the same return as a unit of domestic currency that is invested for one period and then exchanged; so that there is no riskless profit. In the following discussion, S follows a discrete Brownian

Motion *i.e* a binary (or tertiary) random walk.

The drawback of this method is the unrealistic assumption that US_i is considered fixed, which is not the real case. In this paper, we allow US_i to vary and consider two distinct (however not uncorrelated) stochastic processes corresponding to the two interest rates and then obtain a new Black-Scholes formula. The following step is to use an explicit finite difference method to obtain the discrete decision tree.

1.2 A Black and Scholes formula with Varying Interest Rates

The idea of using time varying interest rates has already been used in Heath-Jarrow-Morton [1987], and in Grabbe [1983]. In this paper we use Grabbe's technique in order to derive a new Black-Scholes formula and later, to state the partial derivative equations characterizing the price of a call option.

Grabbe develops a pricing relationship both for American and European call options on foreign currency. We shall concentrate our discussion upon the latter. Indeed unlike European call options, American call options do

not satisfy an equation, but they do satisfy a number of inequalities.

What is more, as we shall see later, American put options are not related to call options through a parity equality, but again through an inequality. This fact is true even for standard options and has been developed in Cox-Rubinstein [1985].

Our development is based on Grabbe's notation : As before, $S(t)$ will be the spot domestic currency price of a unit of foreign exchange at time t , $c(t)$ is the domestic currency price at time t of an European call option written on one unit of foreign exchange, X is the domestic currency exercise price of an option on foreign currency, and finally, $B(t, T)$ and $B^*(t, T)$ are the domestic and foreign currency prices, respectively, of a pure discount bond which pays one unit of the corresponding currency at time $t + T$. These latter quantities are:

$$B(t, T) = e^{-USiT} \quad (3)$$

$$B^*(t, T) = e^{-FCiT} \quad (4)$$

But again, we no longer consider these interest rates as fixed anymore. Therefore, we consider three Brownian motions, namely $S(t)$, $B(t)$, and $B^*(t)$;

$$\frac{dS}{S} = \mu_S(t)dt + \sigma_S(t)dx \quad (5)$$

$$\frac{dB}{B} = \mu_B(t, T)dt + \sigma_B(t, T)dz \quad (6)$$

$$\frac{dB^*}{B^*} = \mu_{B^*}(t, T)dt + \sigma_{B^*}(t, T)dy \quad (7)$$

where dx, dy, dz are geometrical Brownian motions, μ corresponds to the mean value and σ to the standard deviation of each process.

What is more, these Brownian motions are correlated and have the following correlation coefficients: $\rho_{SB}, \rho_{SB^*}, \rho_{BB^*}$. It is more convenient to use two random processes $B(t)$ and

$$G(t) = SB^*(t)$$

and $G(t)$ will satisfy

$$\frac{dG}{G} = \mu_G(t, T)dt + \sigma_G(t, T)dw \quad (8)$$

$$\mu_G = \mu_S + \mu_{B^*} + \rho_{SB^*} \sigma_S \sigma_{B^*}$$

$$\sigma_G dw = \sigma_S dx + \sigma_{B^*} dy$$

and we shall have one correlation coefficient ρ_{GB^*} .

According to Grabbe, an European call option will have to satisfy boundary conditions that we do not discuss here, since it is not relevant to our problem. However it is important to underline again the fact that American call options satisfy a particular differential inequality , and have boundary conditions to verify.

Now, using Ito's lemma we find:

$$dc = \frac{\partial c}{\partial G} dG + \frac{\partial c}{\partial B} dB + \frac{\partial c}{\partial T} dT - \frac{1}{2} \Gamma dT \quad (9)$$

$$dT = -dt$$

$$\Gamma = \frac{\partial^2 c}{\partial G^2} G^2 \sigma_G^2 + 2 \frac{\partial^2 c}{\partial G \partial B} GB \rho_{GB} \sigma_G \sigma_B + \frac{\partial^2 c}{\partial B^2} B^2 \sigma_B^2$$

Using a non arbitrage principle Grabbe shows that the equation (9) is equivalent to the equation

$$\frac{\partial c}{\partial t} = -\frac{1}{2}\Gamma \quad (10)$$

Finally, using this equation and the boundary conditions for an European call option, and *considering CONSTANT volatilities and a CONSTANT correlation coefficient* we have the Black Scholes formula:

$$c(t) = S(t)B^*(t, T)\mathcal{N}(d_1) - XB(t, T)\mathcal{N}(d_2) \quad (11)$$

where again \mathcal{N} is the cumulative normal distribution with mean zero and variance one, and

$$d_1 = \frac{\ln(G/XB) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\sigma^2 = \sigma_G^2 + \sigma_B^2 - 2\rho_{GB}\sigma_G\sigma_B$$

the last equation is a simplified version of the general case in Grabbe's paper where volatilities and the correlation coefficient are time dependent.

Therefore while in the constant interest case we had to choose a fixed V and a fixed USi , here we choose σ_B , σ_G , ρ_{GB} and we use two stochastic processes G and B instead of one in the previous case. Needless to say, the time varying interest rate case is a generalization of the constant interest case and if we choose B and B^* as constant, we would have the same Black Scholes formula and $\sigma = \sigma_S = V$.

1.3 Application of the Explicit Finite Difference Method

This section will use the explicit finite difference method technique as developed in Hull-White [1990].

First we shall explain the meaning of the word *explicit*. Indeed there are two ways of implementing the finite difference approach. The explicit finite difference method, relates the value of the derivative security at time t to

other alternative values at time $t + \Delta t$. The implicit finite difference method, relates the value of the derivative security at time $t + \Delta t$ to other alternative values at time t . Both these methods are equivalent to multinomial lattice approaches, but for the implicit method, in the limit, the underlying variable can move from its value at time t to an infinity of possible values at time $t + \Delta t$. Brennan and Schwartz [1978] proved these two points thoroughly.

Note that the Lattice approach, which corresponds to starting with a discrete tree, is another popular procedure for valuing options. This method has been developed by Cox, Ross, Rubinstein [1979]. There is an interesting comparison between lattice and finite difference approaches. It can be proved (*cf* Geske and Shastri [1985]) that the explicit finite difference method, with logarithmic transformations, is the most efficient approach when large numbers of stock options are being evaluated.

One of the principal advantages of the explicit finite difference method is that it is computationally much simpler than the implicit method since it does not require the inversion of matrices. It is conceptionally simpler since it is much more closely related to the lattice approach and finally it requires

fewer boundary conditions than the implicit method.

Hull and White illustrate the method by valuing bonds and bond options when interest rates are described by the Cox, Ingersoll, Ross model. They consider a derivative security with price f that depends on a single stochastic variable θ and the stochastic equation followed by θ is:

$$d\theta = \mu(\theta, t)\theta dt + \sigma(\theta, t)\theta dz$$

where dz is a Wiener process. They call λ the market price of risk of θ and use a proof by Garman [1976] to show that f must satisfy the differential equation:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \theta}(\mu - \lambda\sigma)\theta + \frac{1}{2}\theta^2\sigma^2\frac{\partial^2 f}{\partial \theta^2} = rf$$

where r is the risk free interest rate. Then using the same kind of approximation that we are going to use in our case, they end up with a trinomial lattice. But in their case, not including the time variable, there is only *one* variable, namely θ ; while in our case, as we will see we shall have two different discrete variables.

Hull and White also suggest using $\ln \theta$ rather than θ as the underlying variable when the finite difference method is applied. We shall use the same transformation in our variables. As they say this transformation is convenient since when σ , the volatility is constant the instantaneous standard deviation of $\ln \theta$ is constant. Thus, the standard deviation of changes in $\ln \theta$ in a time interval Δt is independent of θ and t . Finally, Hull and White suggest that in case of dealing with two variables (which is our case), we transform them to two uncorrelated variables. In order to eliminate the correlation they suggest a rotation in our coordinates. This is indeed what we are going to do.

As we said in the previous section, we use two random processes B and G . Their volatilities are not constant but rather proportionnal to them, and in order to obtain random processes with constant volatilities we are going to use:

$$\tilde{G} = \ln G \tag{12}$$

$$\tilde{B} = \ln B \quad (13)$$

and the stochastic processes \tilde{B} and \tilde{G} do have constant variances, more precisely we have:

$$d\tilde{G} = \mu_{\tilde{G}}dt + \sigma_{\tilde{G}}dw \quad (14)$$

$$d\tilde{B} = \mu_{\tilde{B}}dt + \sigma_{\tilde{B}}dz \quad (15)$$

and what is more

$$\sigma_{\tilde{G}} = \sigma_G = \textit{constant}$$

$$\sigma_{\tilde{B}} = \sigma_B = \textit{constant}$$

Note that there is an easy financial interpretation of these processes: \tilde{B} is simply $-T \times USi$, and \tilde{G} is $-T \times FCi + \ln S$, where $\ln S$ can be seen as the instantaneous return rate of S the exchange rate.

Now having two constant-variance processes, the next step will be to eliminate the linear dependence between them, in other words using a technique suggested by Hull and White, we are going to obtain two *UNCORRELATED* processes. This can be done by a rotation of our variables; more precisely:

$$\Phi = \sigma_B \tilde{G} + \sigma_G \tilde{B} \quad (16)$$

$$\Psi = \sigma_B \tilde{G} - \sigma_G \tilde{B} \quad (17)$$

and as we said Φ and Ψ will be uncorrelated.

What is more

$$\sigma_\Phi = \sigma_B \sigma_G (2(1 + \rho_{GB}))^{\frac{1}{2}}$$

$$\sigma_\Psi = \sigma_B \sigma_G (2(1 - \rho_{GB}))^{\frac{1}{2}}$$

Of course it is possible to invert the linear transformation we have used, to obtain \tilde{G} and \tilde{B} as functions of Φ and Ψ :

$$\tilde{G} = \frac{\Phi + \Psi}{2\sigma_B}$$

$$\tilde{B} = \frac{\Phi - \Psi}{2\sigma_G}$$

and obviously $B = \exp(\tilde{B})$ and $G = \exp(\tilde{G})$. So given the values of Φ and Ψ , we can obtain G and B easily.

Note that unlike \tilde{B} and \tilde{G} , there is no obvious financial interpretation for Φ and Ψ . The best we can say is that they are two uncorrelated linear combinations of the domestic interest rate, the foreign interest rate and the logarithm of the exchange rate.

Now having these two constant-variance and uncorrelated stochastic processes, we are going to use them in the differential equation (10) and using

$$\frac{\partial c}{\partial G} = \frac{\partial c}{\partial \Phi} \frac{\partial \Phi}{\partial G} + \frac{\partial c}{\partial \Psi} \frac{\partial \Psi}{\partial G}$$

and

$$\frac{\partial \Phi}{\partial G} = \frac{\sigma_B}{G}$$

$$\frac{\partial \Psi}{\partial G} = \frac{\sigma_B}{G}$$

we shall have

$$\frac{\partial c}{\partial G} = \frac{\sigma_B}{G} \left(\frac{\partial c}{\partial \Phi} + \frac{\partial c}{\partial \Psi} \right)$$

and similarly

$$\frac{\partial c}{\partial B} = \frac{\sigma_G}{B} \left(\frac{\partial c}{\partial \Phi} - \frac{\partial c}{\partial \Psi} \right)$$

Repeating the same operation, we will find the two second order derivatives and the second order cross derivative and finally:

$$\frac{\partial^2 c}{\partial G^2} G^2 \sigma_G^2 = \sigma_B^2 \sigma_G^2 \left(\frac{\partial^2 c}{\partial \Phi^2} + \frac{\partial^2 c}{\partial \Psi^2} + 2 \frac{\partial^2 c}{\partial \Phi \partial \Psi} \right)$$

$\frac{\partial^2 c}{\partial B^2} B^2 \sigma_B^2$ will be exactly the same except for a minus sign before the cross derivative term, therefore the sum of the two expressions above will not contain any cross derivative term, and finally

$$2\rho_{GB}\sigma_G\sigma_B GB \frac{\partial^2 c}{\partial G \partial B} = 2\sigma_B^2 \sigma_G^2 \rho_{GB} \left(\frac{\partial^2 c}{\partial \Phi^2} - \frac{\partial^2 c}{\partial \Psi^2} \right)$$

Now using all these expressions in equation (10) we can observe that no cross derivative will remain in the equation, which is logical since Φ and Ψ are uncorrelated; the equation obtained is

$$\frac{\partial c}{\partial t} = -\sigma_B^2 \sigma_G^2 [(1 + \rho_{GB}) \frac{\partial^2 c}{\partial \Phi^2} + (1 - \rho_{GB}) \frac{\partial^2 c}{\partial \Psi^2}] \quad (18)$$

So we have our new differential equation with variables Φ and Ψ .

Now we are prepared to use Hull and White's explicit finite difference method. In order to do this, we shall define three discrete variables:

$$t_i = t_0 + i\Delta t \quad (19)$$

$$\Phi_j = \Phi_0 + j\Delta\Phi$$

$$\Psi_k = \Psi_0 + k\Delta\Psi$$

where t_0 is the initial time, Φ_0 is the initial value of the variable Φ and the same for Ψ_0 . Note that the three integers i , j , and k vary independently and for a given stage i any value of Φ and Ψ are possible. Once again our two random processes are linearly independent and therefore could vary independently from one another.

Now the discretization will give us, calling $c(i, j, k)$ the price of our call option for $t = t_i$, $\Phi = \Phi_j$ and $\Psi = \Psi_k$;

$$\frac{\partial^2 c}{\partial \Phi^2} \simeq \frac{c(i, j+1, k) + c(i, j-1, k) - 2c(i, j, k)}{2\Delta\Phi^2}$$

$$\frac{\partial^2 c}{\partial \Psi^2} \simeq \frac{c(i, j, k+1) + c(i, j, k-1) - 2c(i, j, k)}{2\Delta\Psi^2}$$

$$\frac{\partial c}{\partial t} \simeq \frac{c(i, j, k) - c(i-1, j, k)}{\Delta t}$$

Now we use these discretized values in our continuous equation (18) and we obtain the new Discrete equation:

$$\begin{aligned}
c(i-1, j, k) = & a(j, k)c(i, j, k) + a(j+1, k)c(i, j+1, k) + a(j-1, k)c(i, j-1, k) \\
& + a(j, k+1)c(i, j, k+1) + a(j, k-1)c(i, j, k-1)
\end{aligned} \tag{20}$$

with

$$\begin{aligned}
a(j, k) = & 1 - \Delta t \sigma_B^2 \sigma_G^2 \left[\frac{1 + \rho_{GB}}{\Delta \Phi^2} + \frac{1 - \rho_{GB}}{\Delta \Psi^2} \right] \\
a(j+1, k) = & a(j-1, k) = \Delta t \sigma_B^2 \sigma_G^2 \frac{1 + \rho_{GB}}{2\Delta \Phi^2} \\
a(j, k+1) = & a(j, k-1) = \Delta t \sigma_B^2 \sigma_G^2 \frac{1 - \rho_{GB}}{2\Delta \Psi^2}
\end{aligned}$$

Moreover,

$$a(j, k) + a(j+1, k) + a(j-1, k) + a(j, k+1) + a(j, k-1) = 1$$

so $a(j, k)$'s could be considered as *Transition Probabilities* in time from $i - 1$ to i . Note that these transition probabilities do not depend on the stage, so we have a stationary system. Consequently, we have one horizontal transition probability, namely $a(i, j)$, an upward transition probability for Φ : $a(j + 1, k)$ etc ... But the computation of $c(i, j, k)$ is done backward in time by computing its expectation on every possible state at the following stage.

Note that normally each of the processes Φ and Ψ corresponding to a trinomial decision tree, we should have expected a 9-ary tree *ie* the cartesian product of two trinomial trees. However, here we have a 5-ary tree, which diminishes the computations considerably. This is due to the fact that Φ and Ψ being uncorrelated, their variations are decoupled, which means that when Φ varies upward or downward, Ψ does not vary, and *vice versa*.

What is more, since for each process the upward transition probability is equal to the downward transition probability, we can put them together :

$$c(i - 1, j, k) = a(j, k)c(i, j, k) + a(j', k)c(i, j', k) + a(j, k')c(i, j, k') \quad (21)$$

with $j' \in \{j+1, j-1\}$ and $k' \in \{k+1, k-1\}$

and

$$a(j', k) = 2a(j+1, k) = 2a(j-1, k)$$

$$a(j, k') = 2a(j, k+1) = 2a(j, k-1)$$

so

$$a(j, k) + a(j', k) + a(j, k') = 1$$

So with these notations we shall have a trinomial decision tree, which, once again reduces the number of computations to calculate the expectation enormously.

However, doing this, we shall lose the information concerning the direction (upward or downward) of our transition. So depending on the number of stages we can choose to use the more precise but more complex 5-ary tree or the less precise but easier trinomial tree.

In our implementation section we shall keep both upward and downward transitions, but we shall choose our parameters such that the horizontal

transition will have a zero probability so that we shall have only 4 branches.

1.4 Conclusion

We have developed first a continuous and then a discrete model for the evaluation of the prices of call options on currencies with time varying interest rates. It is important to note that the discrete model did not come first as for example in Cox-Rubinstein [1985], but it was rather obtained artificially, by discretizing the continuous differential equation.

The other important fact to underline is that in order to have our transition probabilities, we have to choose our time and processes variation intervals, but also choose two volatilities and a correlation coefficient. These three last quantities could be estimated by identification with real data.

2 Implementation

2.1 The Parameters

As we saw previously, we have a decision tree that can be made 4-ary by choosing convenient parameters. More precisely we would like to set the horizontal transition probability to zero. This latter is equal to:

$$a(j, k) = 1 - \Delta t \sigma_B^2 \sigma_G^2 \left[\frac{1 + \rho_{GB}}{\Delta \Phi^2} + \frac{1 - \rho_{GB}}{\Delta \Psi^2} \right]$$

The best would be therefore to choose our stochastic as well as the financial parameters and then deduce the time interval. Once these parameters determined all transition probabilities will be fixed. But practically, the program user does not have access to the abstract parameters as Φ_0 , Ψ_0 , $\Delta \Phi$ or $\Delta \Psi$. Consequently, it is more logical to let the user choose the real financial parameters as the initial (present) exchange rate S_0 , the present domestic interest rate B_0 , the present foreign interest rate B_0^* and also three variation steps ΔS , ΔB and ΔB^* .

Once these choices are done, the program can evaluate the *abstract* pa-

rameters by using the relations:

$$\Phi_0 = \sigma_B \ln(G_0) + \sigma_G \ln(B_0)$$

$$\Psi_0 = \sigma_B \ln(G_0) - \sigma_G \ln(B_0)$$

The equations written above are *exact* but in order to estimate $\Delta\Phi$ and $\Delta\Psi$ an approximation must be made, namely, we suppose that the exchange rate and the foreign interest rate make no brutal variations from their present value so that $\Delta G = \Delta(SB^*) \simeq \Delta SB_0^* + \Delta B^* S_0$. What is more we can also use the linear approximation $\Delta \ln(x) \simeq \frac{\Delta x}{x}$ which will give us the following relations:

$$\Delta\Phi \simeq \sigma_B \frac{\Delta SB_0^* + \Delta B^* S_0}{S_0 B_0^*} + \sigma_G \frac{\Delta B}{B}$$

$$\Delta\Psi \simeq \sigma_B \frac{\Delta SB_0^* + \Delta B^* S_0}{S_0 B_0^*} - \sigma_G \frac{\Delta B}{B}$$

Now, having our parameters, the decision tree can be constructed. The

backward induction relation is completely known and creates no difficulty but there remains though the determination of the final (Boundary) values. Let us suppose that we have determined the final values of Φ and Ψ , that is to say their values on the expiration day. How could we deduce the value of our call option? For this, we should first determine the value of S , the exchange rate on the expiration day, and then deduce the value of the call by using the well known relation $c = \max(0, S - K)$ where K is the striking price.

But the determination of the final value of S , having the final values of Φ and Ψ is not obvious. Indeed we can easily have the final values for G and B using:

$$G = \exp\left(\frac{\Phi + \Psi}{2\sigma_B}\right)$$

$$B = \exp\left(\frac{\Phi - \Psi}{2\sigma_G}\right)$$

but again since $G = SB^*$, how could we deduce S ? One idea is to use the no arbitrage relation at the expiration date, which means: $(1 + B)S_T = S_{T+1}(1 + B^*)$ but we do not have access to the value of S at $T + 1$ so we could take the expectation or suppose that after the expiration date the value of S

will not change and approximately $S_T = S_{T+1}$.

Finally, we could also let the user choose, not the Bond Values B and B^* used in Grabbe's paper, but the real initial interest values US and FC as was done in the paper Klaassen-Shapiro-Spitz. In this case we shall have

$$B_0 = \exp(-TUS_0)$$

and

$$B_0^* = \exp(-TFC_0)$$

and again with an approximation on little variations:

$$dB = -TdUS \exp(-TUS_0)$$

and

$$dB^* = -TdFC \exp(-TFC_0)$$

2.2 The Decision Tree

As we have already said, we have a 4-ary tree. We must *identify* each node by labeling it. For this purpose we shall choose the following convention. The first node corresponding at the present time $t = 1$ will be numerated 0. For this node the time i is also equal to zero and our two variables Φ and Ψ , take their initial values Φ_0 and Ψ_0 . This means that by writing for each node

$$\Phi = \Phi_0 + j\Delta\Phi$$

and

$$\Psi = \Psi_0 + k\Delta\Psi$$

each node will correspond to a *non unique* value of the pair (j, k) . Obviously, the first node possesses a pair $(j, k) = (0, 0)$.

Now, the question is to what pair each node at time i corresponds? First it is clear that at time i , we have 4^i nodes. If we subtract the sum $4^0 + 4^1 + 4^2 + \dots + 4^{i-1}$ of the number affected with our convention to each node, the numbers of the nodes will belong to the interval $\{0, 1, 2, \dots, 4^i - 1\}$. The

sons of the node (j, k) are RESPECTIVELY $(j + 1, k)$, $(j - 1, k)$, $(j, k + 1)$, $(j, k - 1)$.

Let us suppose that the number of the nodes having the value (j, k) is n and let us call $h = n - (4^0 + \dots + 4^{i-1})$ (Of course for $i = 1$ we have $h = n - 1$); It is easy to write $h \in \{0, \dots, 4^{i-1} - 1\}$ in the basis of Four. In other words we divide h by 4 and take the quotient and again we devide the latter by four, until the quotient is inferior to 4, and then taking the remainders of the divisions, as well as the last quotient, we shall have an expression of the form

$$h = h_{i-2}4^{i-2} + \dots + h_04^0$$

Note that with our conventions at stage one $i = 0$ and thus at stage T we shall have $i = T - 1$. Now, the first son of the node of number h will have a number h' such that the coefficients of the number h in the 4 basis are shifted to left and the last coefficient is $p_1 = 0$. The same statement is valid for the second son with $p_2 = 1$, third $p_3 = 2$ and fourth $p_4 = 3$. So

$$h' = h_{i-2}4^{i-1} + \dots + p_14^0$$

Now we would like to find the real number of the node *first son of n*; So we add again $4^0 + \dots + 4^{i'-1}$ with $i' = i + 1$, to h' and the node number n' will be obtained. But then

$$n' = h' + 4^0 + \dots + 4^i = 4h + p1 + 4^0 + \dots + 4^i = 4(n - [4^0 + \dots + 4^{i-1}]) + p1 + 4^0 + \dots + 4^i$$

and an obvious simplification will give

$$n' = 4n + p1 + 4^0 = 4n + 1$$

and in the same way the second son will have the node number $4n + 2$ and the third $4n + 3$ and the fourth $4n + 4$.

So to sum up our reasoning, if a node of value (j, k) at time i has a node number of n , then its sons of value $(j + 1, k)$, $(j - 1, k)$, $(j, k + 1)$, $(j, k - 1)$, at time $i + 1$ will correspond respectively to the node number $4n + 1$, $4n + 2$, $4n + 3$, and $4n + 4$.

What is more, we saw that the first two sons are related to a transition probability p , and the two others to another probability $q = 1 - p$. We can

thus write our Backward Induction relation on the tree as following:

$$c_i(j, k) = p(c_{i+1}(j + 1, k) + c_{i+1}(j - 1, k)) + q(c_{i+1}(j, k + 1) + c_{i+1}(j, k - 1))$$

or in a much simpler way:

$$c(n) = p(c(4n + 1) + c(4n + 2)) + q(c(4n + 3) + c(4n + 4))$$

with $n \in \{0, \dots, 4^0 + \dots + 4^{T-1} - 1\}$. So now the implementation of the Induction relation is very easy. What is more, given T , we know the exact number of the nodes so we can use an array structure and we do not need to use pointers as it was the case in Klaassen's program on Ho and Lee model [1993].

There remains one major difficulty though, and that is the determination of the final values or the boundary conditions on the Expiration day and this is what we are going to develop in the next section.

2.3 Boundary Values

Having chosen T , as we saw we shall have 4^{T-1} final nodes. Each node will have a number n and a value (j, k) . The question is then: How to find (j, k) , knowing n . Indeed, once we have (j, k) , it is possible to find the final values of Φ and Ψ and then S and finally c for each node n on the expiration day.

Let us take a node $n \in \{4^0 + \dots + 4^{T-2}, \dots, 4^0 + \dots + 4^{T-2} + 4^{T-1} - 1\}$. As in the previous section we shall call $h = n - (4^0 + \dots + 4^{T-2})$ and therefore $h \in \{0, \dots, 4^{T-1} - 1\}$. Having the value of h for a given node, once again, we shall write it in the four basis; more precisely we shall take h divide it by 4 and take the resulting quotient and again... until the quotient becomes inferior to 4. Then we take each remainder of devision *and* the last quotient and check if it is equal to 0, 1, 2, or 3?

We set j and k equal to zero. Each time the answer to the question above is 0 we increase j by one unit, if it is 1 we decrease j by one unit, if it is 2 we increase k by one unit and if it is 3 we decrease k by one unit. There remains though, the problem of the “first” nodes of the expiration day. Indeed, let

us assume $T = 3$ for the first four nodes $h = 0, 1, 2, 3$ and there is no division to be made and thus we have to do a treatment on it, exactly as we did for the last quotient AND increase j *once more*. For example for $h = 0$ we do not have $j = 1$ but $j = 2$! Similarly if $T = 4$ we will have to increase j twice more for $h = 0, 1, 2, 3$ and once more for $h \in \{5, \dots, 15\}$.

It is possible to generalize this process and do the following statement: For $h = 0$ we shall do $j := j + T - 2$ and for each m from 2 to $T - 1$, if $h \in \{4^{T-m-1}, \dots, 4^{T-m} - 1\}$ then $j := j + m - 1$. And thus this special treatment will finish our evaluation of the vales of the nodes on the expiration day.

Note that this process is valid for *any* expiration stage T superior to one as long as the computer capacity is not depassed.

2.4 Computational Experience

Now, we can run the program with given values for different stochastic and financial parameters, and then compare results between different choices of parameters. The main result is of course the present price of the call option.

That is $c[0]$.

What would be still more relevant would be, instead of taking fixed parameters, varying a parameter in a reasonable interval. Then to save the data in a file and use Matlab software to plot the curve representing the price of the call option as a function of this parameter.

The famous curve of the Call Option Price c as a function of S the “underlying Security”, or more precisely, here, the Exchange Rate, has the expected form. This demonstrates the consistency of our model.

Now, having this curve (S, c) we could change other parameters and observe their influence on our function. Since we have done a little variation hypothesis, we shall take each variational interval ΔS , ΔUS and ΔFC equal to ten percent of the initial value S_0 , US_0 , FC_0 . Consequently, the parameters to change are on the one hand the stochastic parameters, namely σ_B , σ_G and ρ_{GB} ; and on the other hand the financial parameters S_0 , US_0 and FC_0 . It is also possible to study the influence of time to expiration T on the curve.

In reality all these influences are foreseeable. We could easily find in financial literature on options the variation direction of the call option price, in

case of an increase of each of the parameters stated above. See for example Cox-Rubinstein [1985]. Indeed, we know that an augmentation in the time until expiration, or equivalently, an increase in the expiration stage T itself, causes the call option price to increase as well, which must be shown in our curve. We shall see that in reality the dependence upon T is more complicated. An increase in the domestic interest rate should raise the call price, which is actually the case in our model. An augmentation in the volatility increases the call price as well.

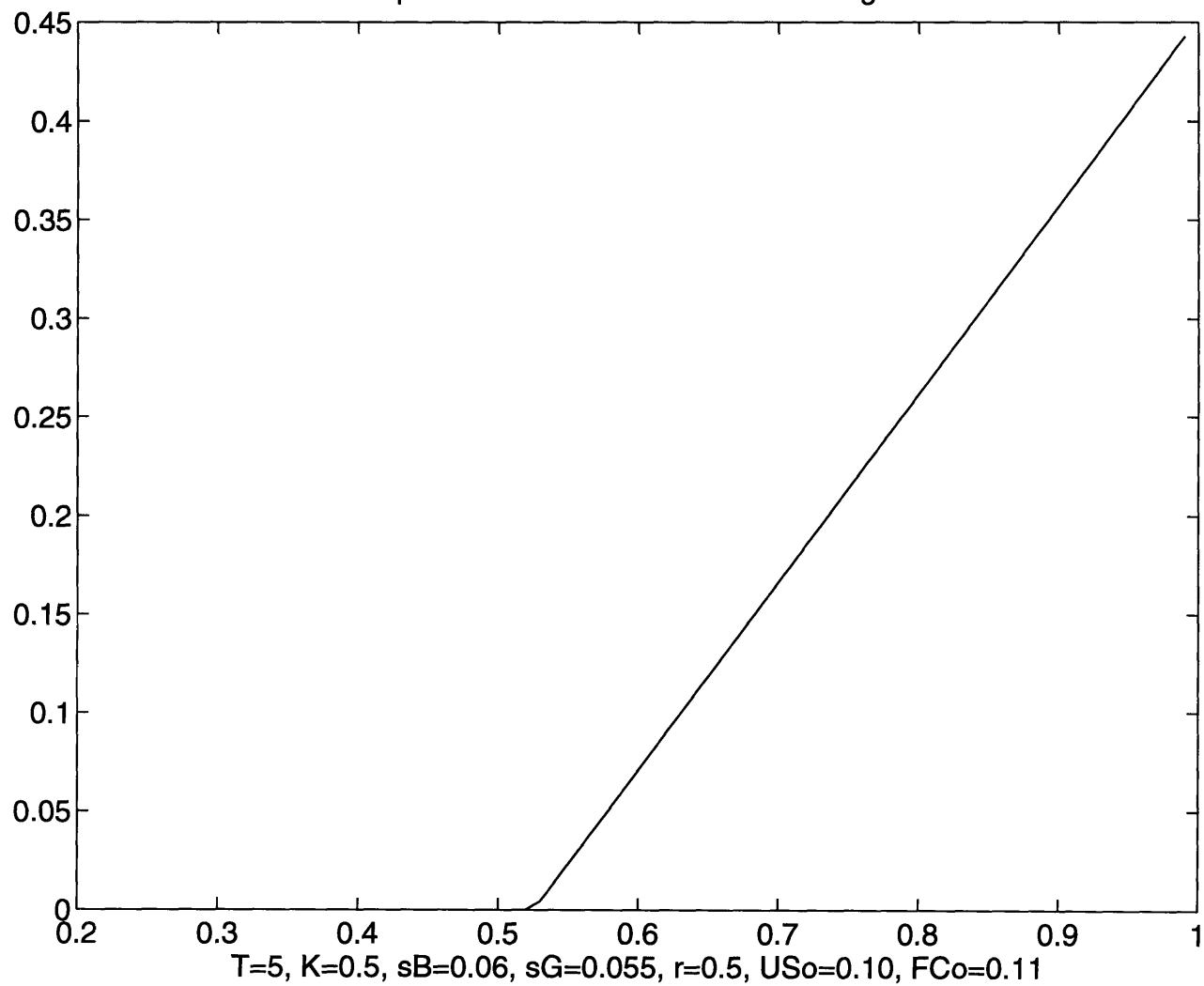
What would also be interesting to do, is to use some real data taken for example in the Klaassen-Shapiro-Spitz paper and observe the results. We could take: $K = 0.5$ and S varying in an interval surrounding this value. since the “composed” volatility V has a value of approximately 0.11 and since we have

$$V^2 = \sigma_B^2 + \sigma_G^2 + 2\rho_{GB}\sigma_B\sigma_G$$

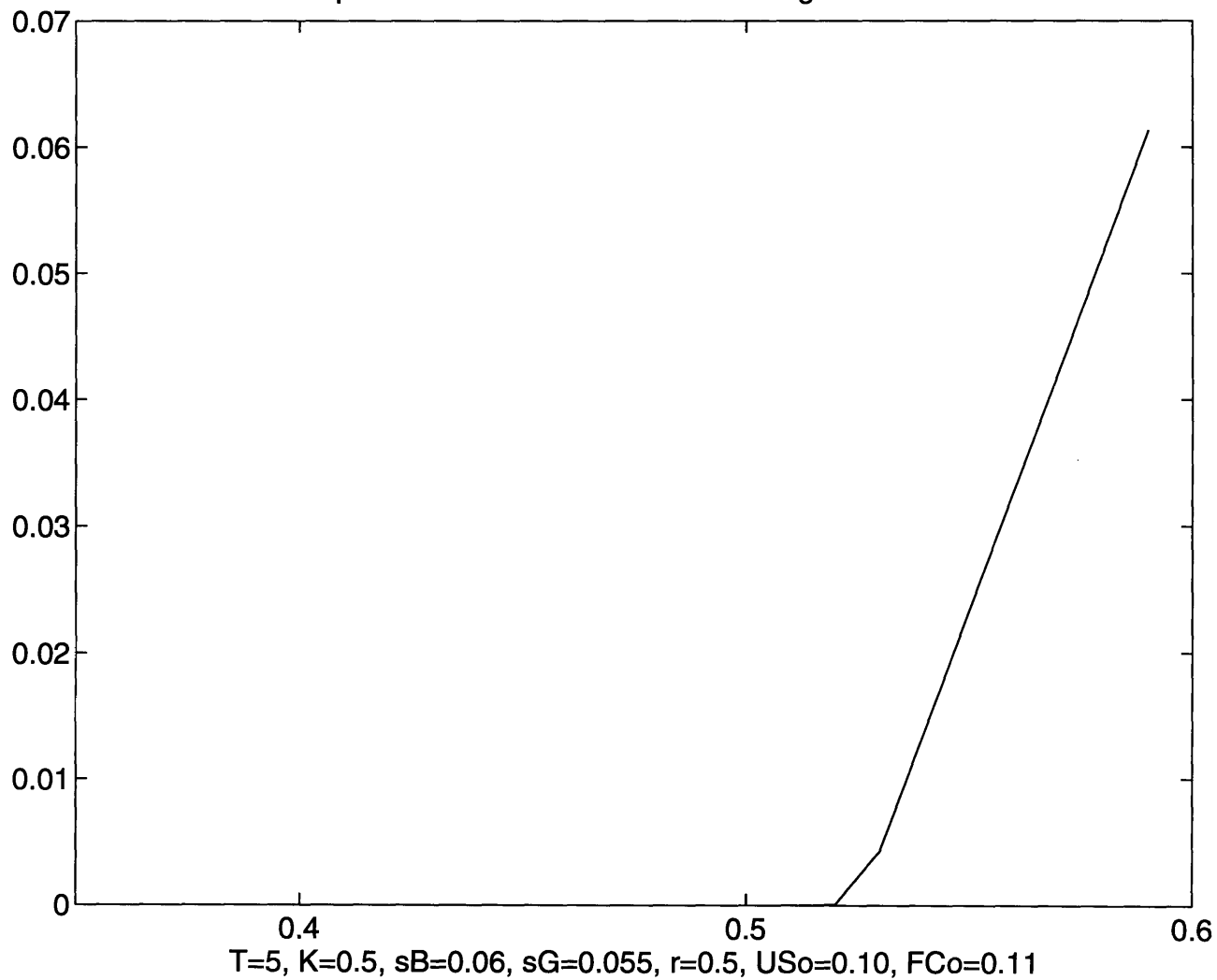
taking ρ_{GB} approximately equal to 0.5, we shall take

$$\sigma_G = \sigma_B = \frac{V}{\sqrt{3}}$$

Call Option Price as a function of Exchange Rate



Call Option Price as a function of Exchange Rate ZOOM



which means a value close to 0.06; As for the initial interest rates we will see that their value could vary and this variation has a great influence on the dependence on *other* parameters.

The fact is that in a simple interest rate model, the expiration date T has only a positive effect on the value of the call option, since the more time available, the more the interest on the Striking price is important and consequently, the more the Call option is valuable. But here, we have TWO interest rates, now if the value of the domestic interest rate is superior to the value of the Foreign interest rate, then the dependence on T will be positive and otherwise this dependence will be reversed. The extreme case will be that of a domestic initial interest rate USo equal to the initial foreign interest rate FCo ; here due to the symmetry of the problem, T will have no influence on the call price and we shall have the same curve for $T = 3$ and $T = 5$.

The financial interpretation of this result is clear: Having the same domestic and foreign interest rate, no matter how far the expiration date is, the call option will have the same value $\max(S - K, 0)$. This is at least true as long as both interest rates have the same volatility. Now what if USo is

initially equal to FCo , but has a larger volatility? It is clear that a higher volatility has an increasing effect on the call option price. But, the results show that this augmentation affects both curves $T = 3$ and $T = 5$ and again there is no sensible positive effect due to T . This is due to the rotation of B and SB^* , providing our two abstract variables Φ and Ψ .

The increase of c due to the volatility, is much more visible with σ_G than σ_B , which is understandable since we have $G = SB^*$ and S represents the underlying security that is to say the *main* variable of the function c .

The break point of the (S, c) curve is $S = K$ if $USo = FCo$, is higher than K if $USo < FCo$ and lower than K otherwise. Again, this result is consistent. In the simple interest rate case, as for instance in Cox-Rubinstein [1985], the break point is always situated before the value of K since of course the domestic interest rate is positive and the foreign interest rate could be taken equal to zero.

The financial interpretation is the following: Suppose we are in the simple interest rate case. On the expiration day we have $c = \max(0, S - K)$ which is represented in the (S, c) plane by a collection of straight lines with

a break point situated exactly at $S = K$. Now, if we are T stages before the expiration day, the value of c will be higher for TWO reasons. First because of the direct effect due to the longer time and therefore higher probability of a change in the value of S , and second due to the interest rate on the value of the striking price that we keep until the expiration day. So these two effects reinforce one another.

In our case, we have two interest rates working in opposite ways: The US interest rate is the *classical* interest rate and it acts exactly as in a single interest rate case. So the expiration time will have a positive effect on c as long as the domestic interest rate is dominant. Indeed it is advantageous to hold the amount of the striking price in Dollars if the interest rate on Dollar is more favorable. So this fact explains both the positive effect of the value of T on c , and the location of the break point of the (S, c) curve before the value $S = K$. The argument will be completely reversed if the Foreign interest rate becomes dominant. This latter situation does not exist in the simple interest rate case.

Other quantities of interest are the sensitivities of the Call as functions

of the underlying security. The first sensitivity to be taken into account is $\frac{\partial c}{\partial \Phi}$ approximated by the absolute value of $\frac{c[1]-c[2]}{2\Delta\Phi}$. The curve has the same form as a $\frac{\partial c}{\partial S}$ curve in a standard case. It is a non decreasing function with a change in its convexity near $S = K$.

The form of the curve is not different for $\frac{\partial c}{\partial \Psi}$; like the previous function it is a non decreasing function. So in this context Φ and Ψ have a symmetric role.

The quantity called Θ in Cox-Rubinstein [1985], is $-\frac{\partial c}{\partial t}$. In our case it could be approximated by the difference of two different values of Calls having the same value of (j, k) , divided by the time interval separating them. Again the result is as expected, a maximum exists around the value of the Striking price.

Another way of studying the properties of the Call option is to plot the curve providing c as a function of other parameters than S . For instance we could plot c as a function of the volatility σ_G . The result is a non decreasing quasi linear function, as in the standard case. But the curve of c as a function of σ_B is different. It is a non monotonous function presenting a minimum at a value a little higher than σ_G . This behaviour is not surprising. Indeed

σ_G represents the *real* volatility, that is the volatility directly related to the underlying security. But σ_B is the volatility of the domestic interest rate, which is completely different. However for values superior to 0.07 c is a non decreasing function of σ_B .

Now if we plot c as a function of US_i we shall have a quasi linear non decreasing function, once again exactly as in the standard case. Needless to say c will be a decreasing function of the foreign interest rate. This function also is quasi linear.

It is also important to study the influence of the correlation between the two interest rates on the value of c . As we had said before, the influence is very weak. c is a decreasing function of ρ_{GB} and its value decreases from 0.14 to 0.10 when ρ_{GB} varies from zero to one. Here again the curve is nearly linear.

2.5 Other Parameters

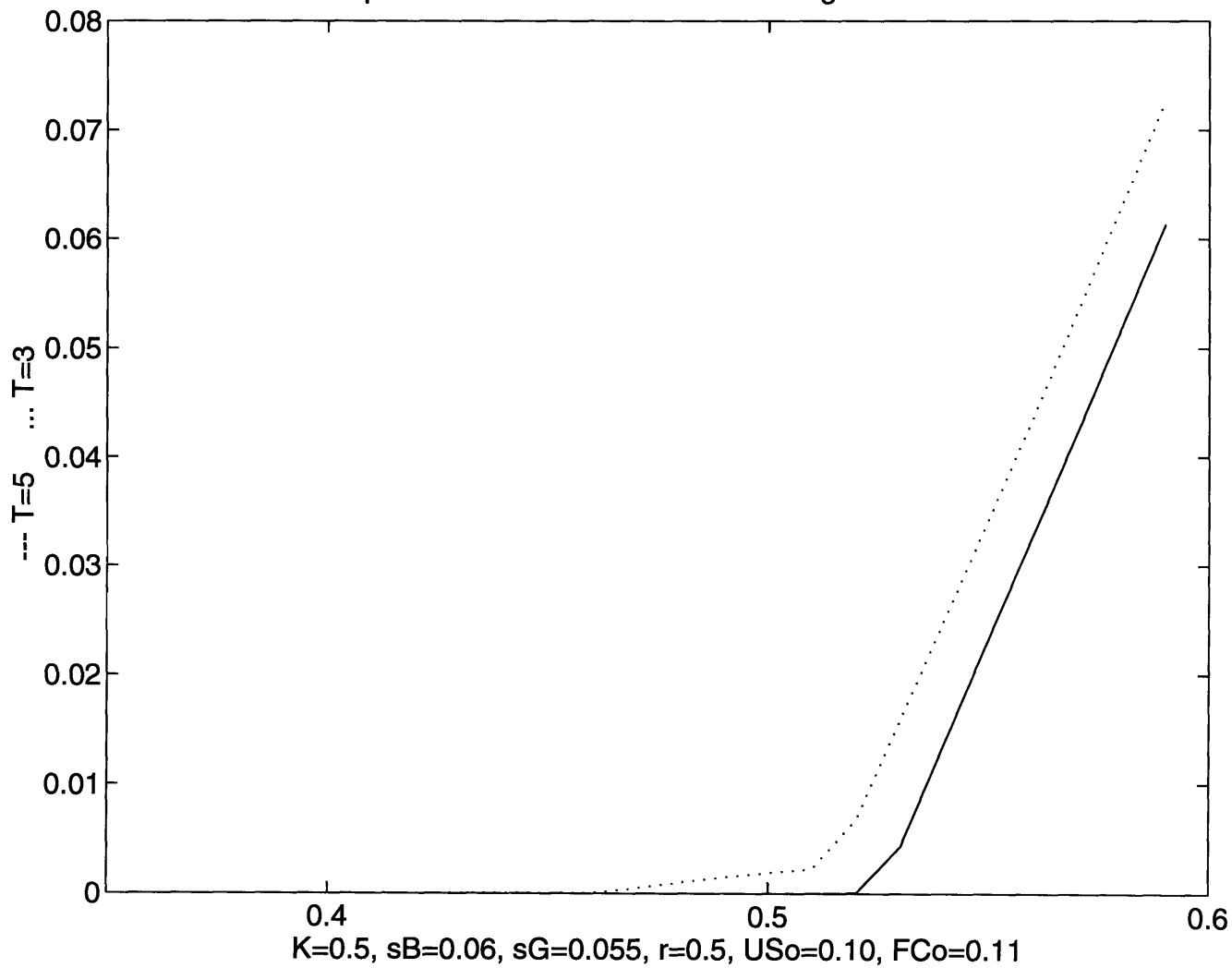
After having studied the influence of the main parameters on the call option price, it would be interesting to see how secondary parameters would influence this value. The well known relation of Put-Call parity should determine the Put option price without ambiguity:

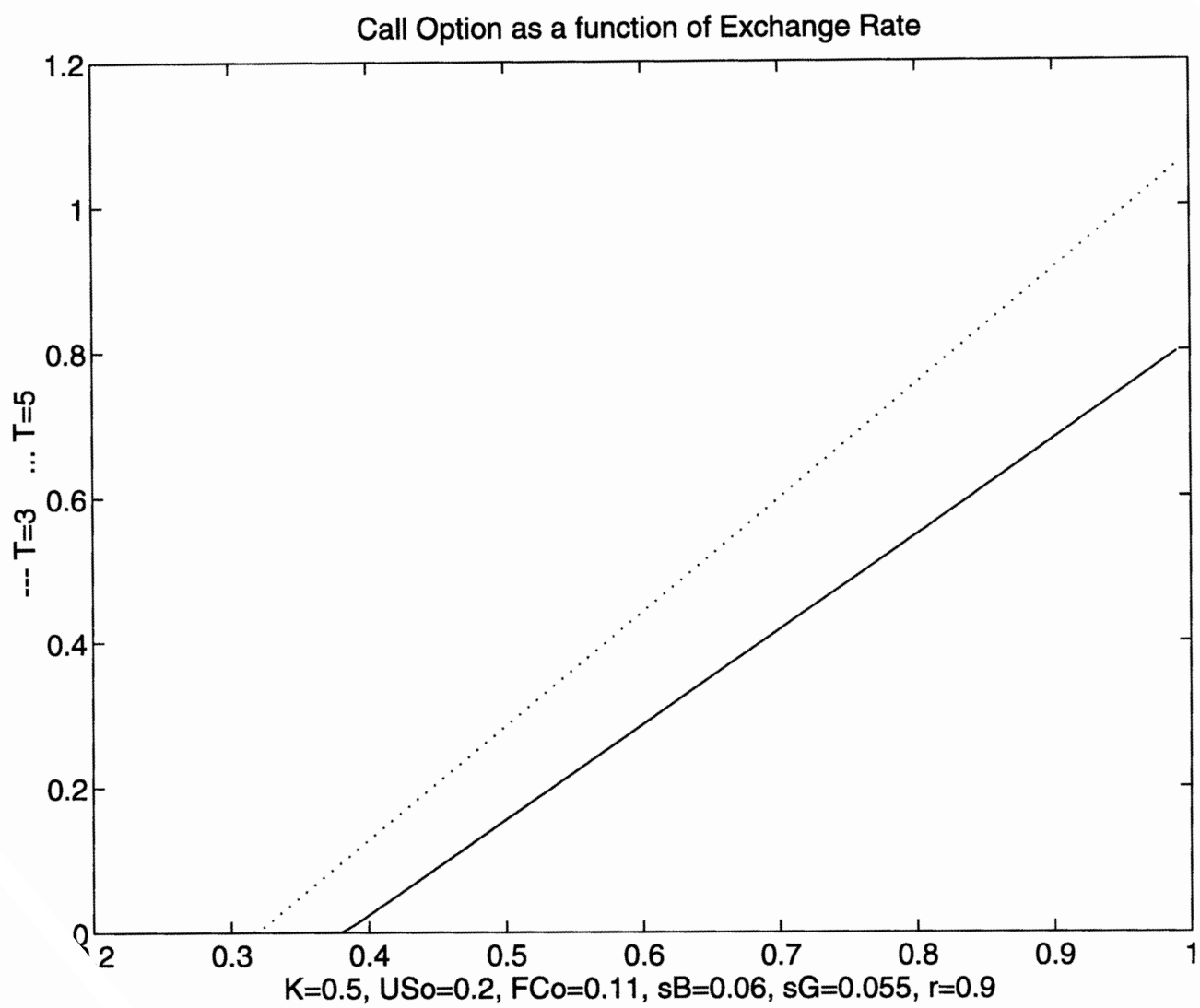
$$C = P + S - K^-$$

where K^- represents the *present* value of the striking price. But what is this present value exactly? If we had only one interest rate r and a time to expiration t then it would be obviously $K^- = Kr^{-t}$. But here, once again we have two different interest rates working in opposite ways. Therefore the present value of the striking price is not necessarily inferior to its absolute value K . Actually, upon a short reflection, this present value would be inferior to the absolute value only if the domestic interest rate USi is superior to the foreign interest rate and more precisely it would be possible to check that:

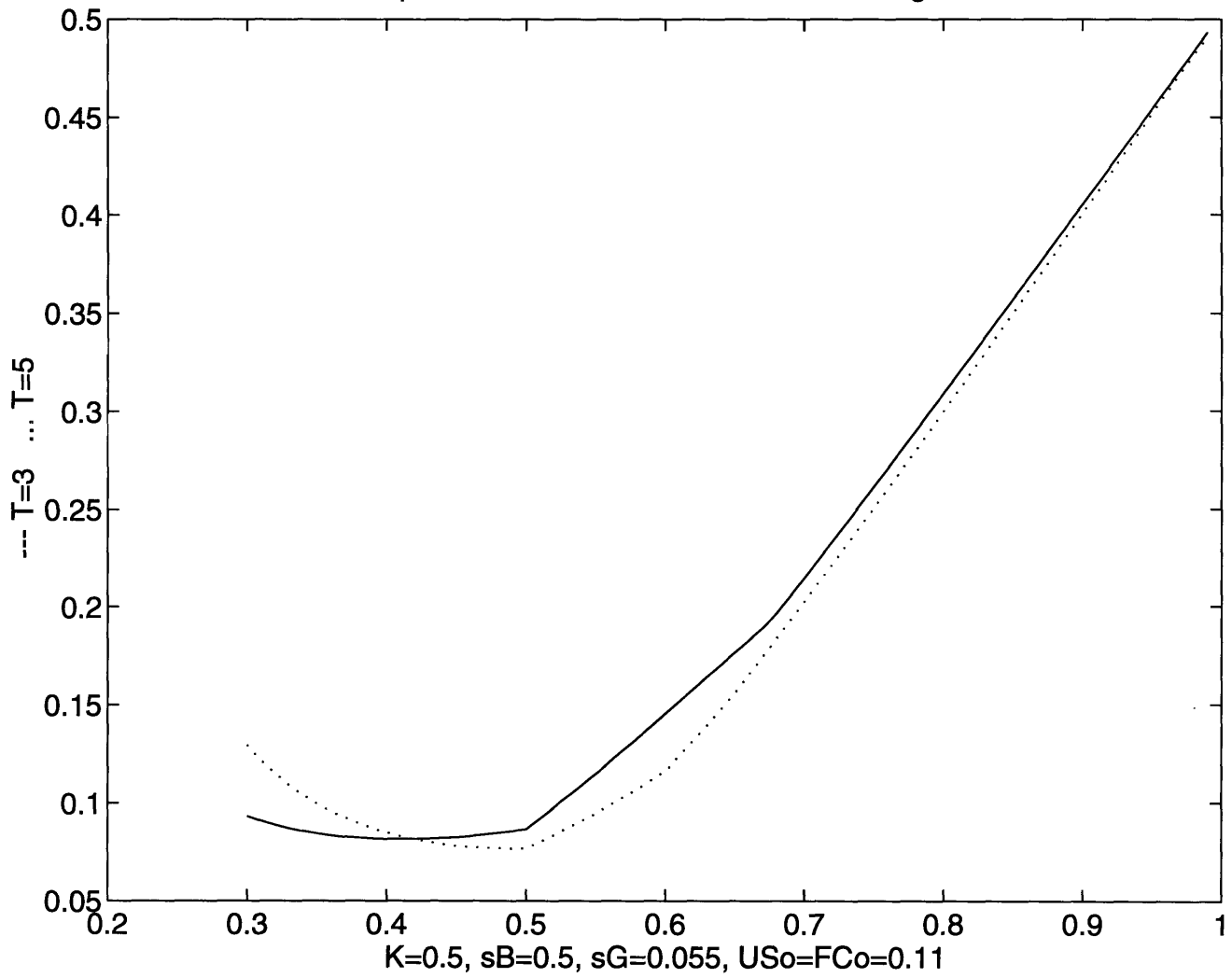
$$K^- = K\left(\frac{US_i}{FC_i}\right)^{-t}$$

Call Option Price as a function of Exchange Rate ZOOM

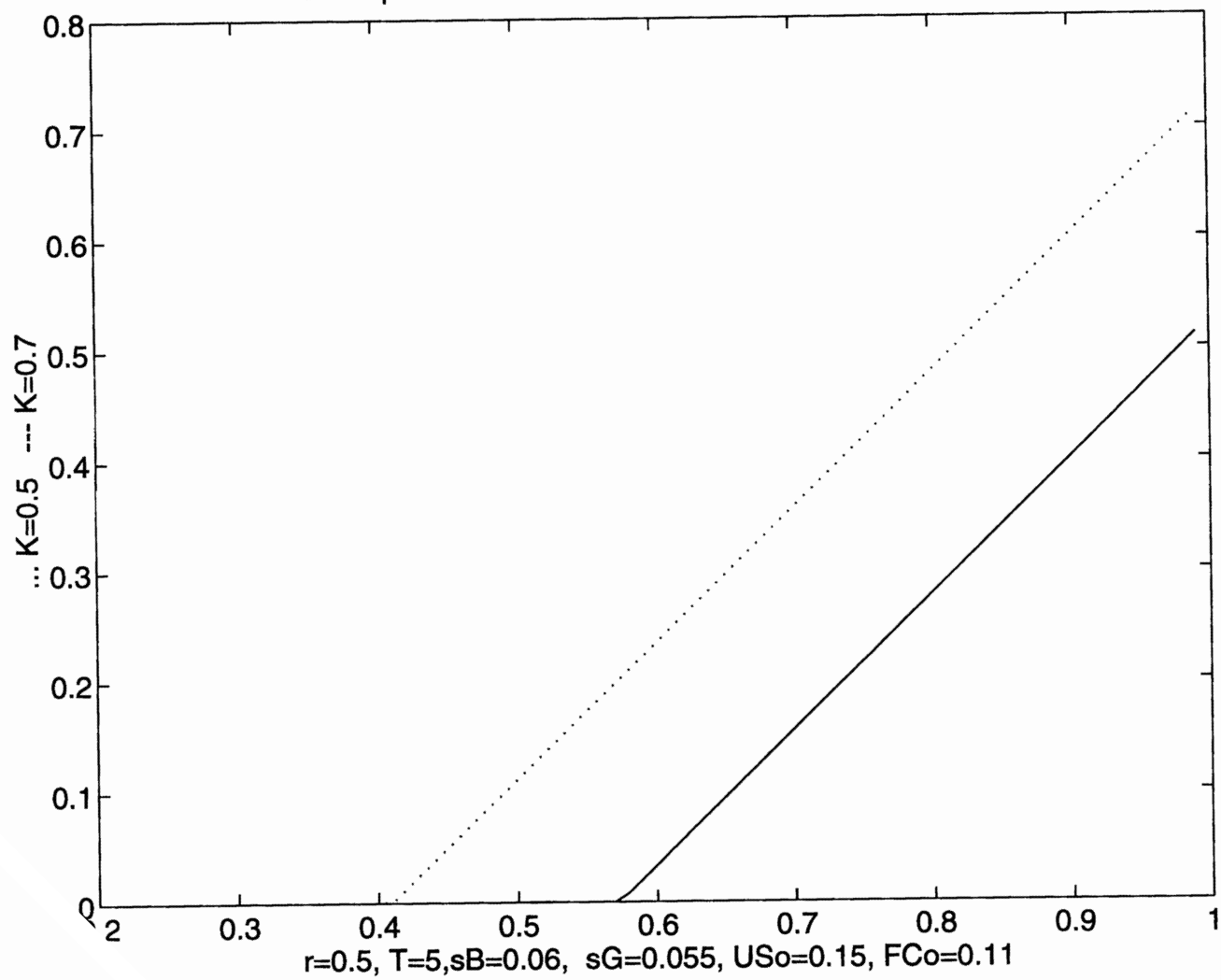




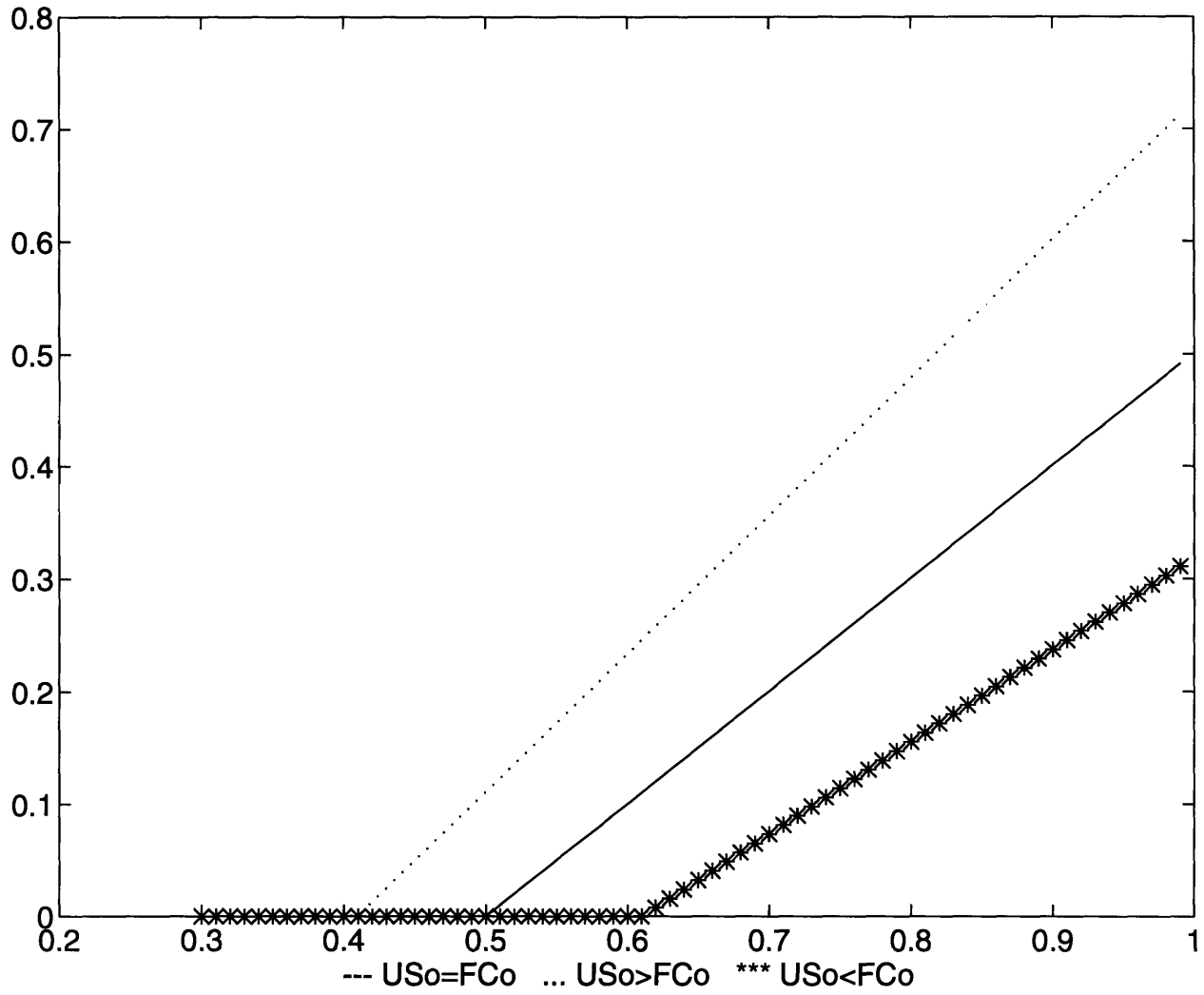
Call Option Price as a function of the Exchange Rate

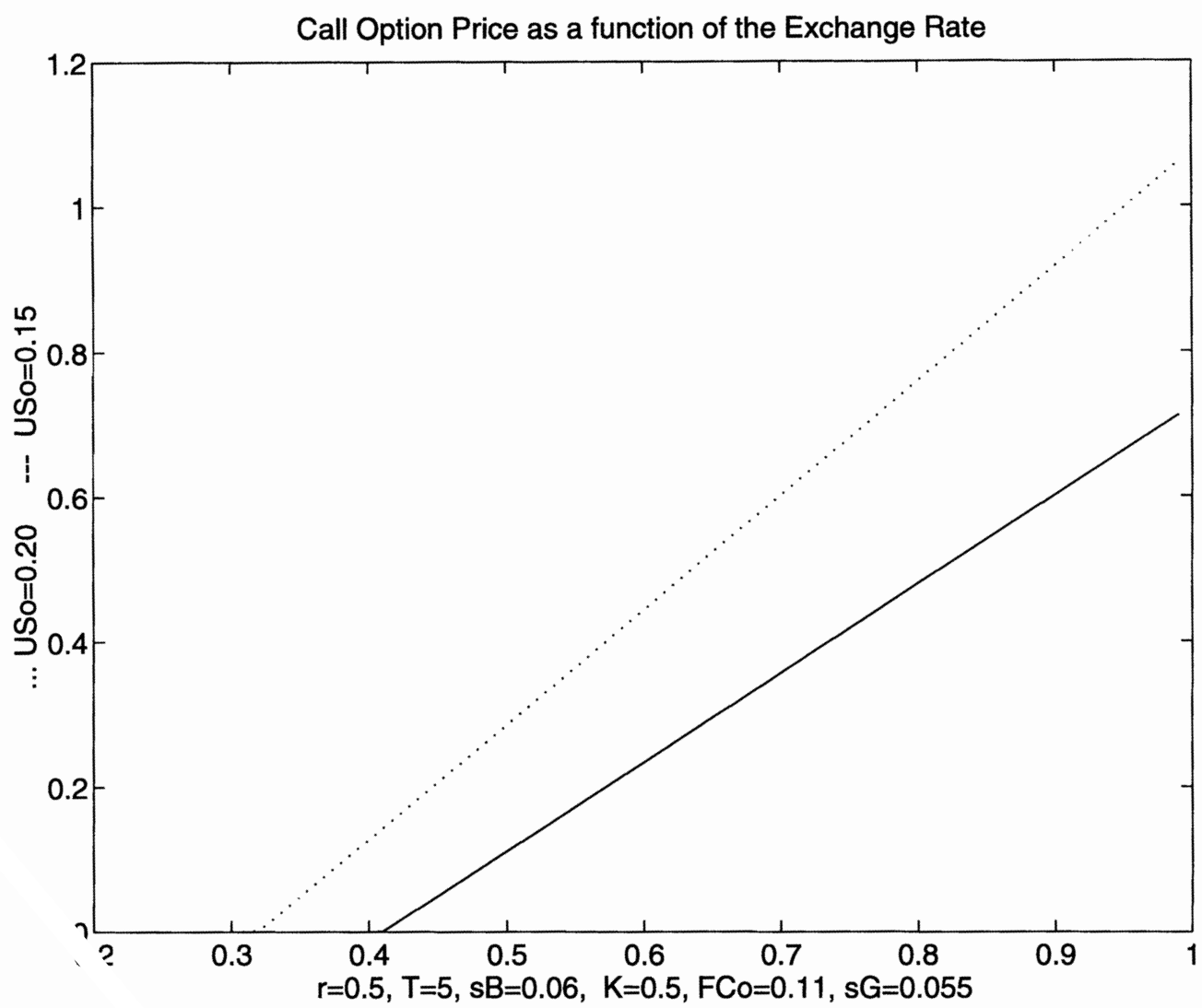


Call Option Price as a function of the Exchange Rate

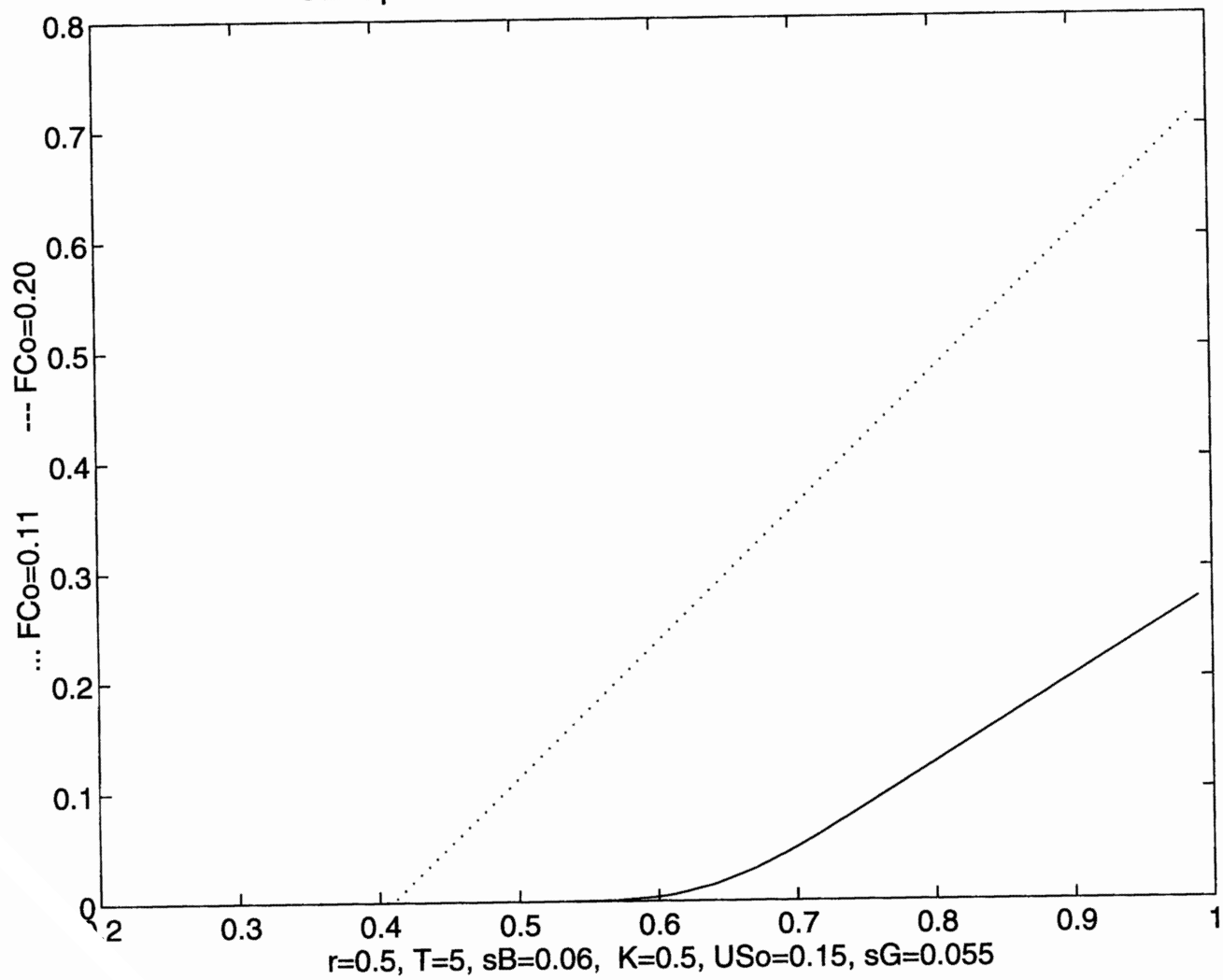


Call Option Price as a function of the Exchange Rate

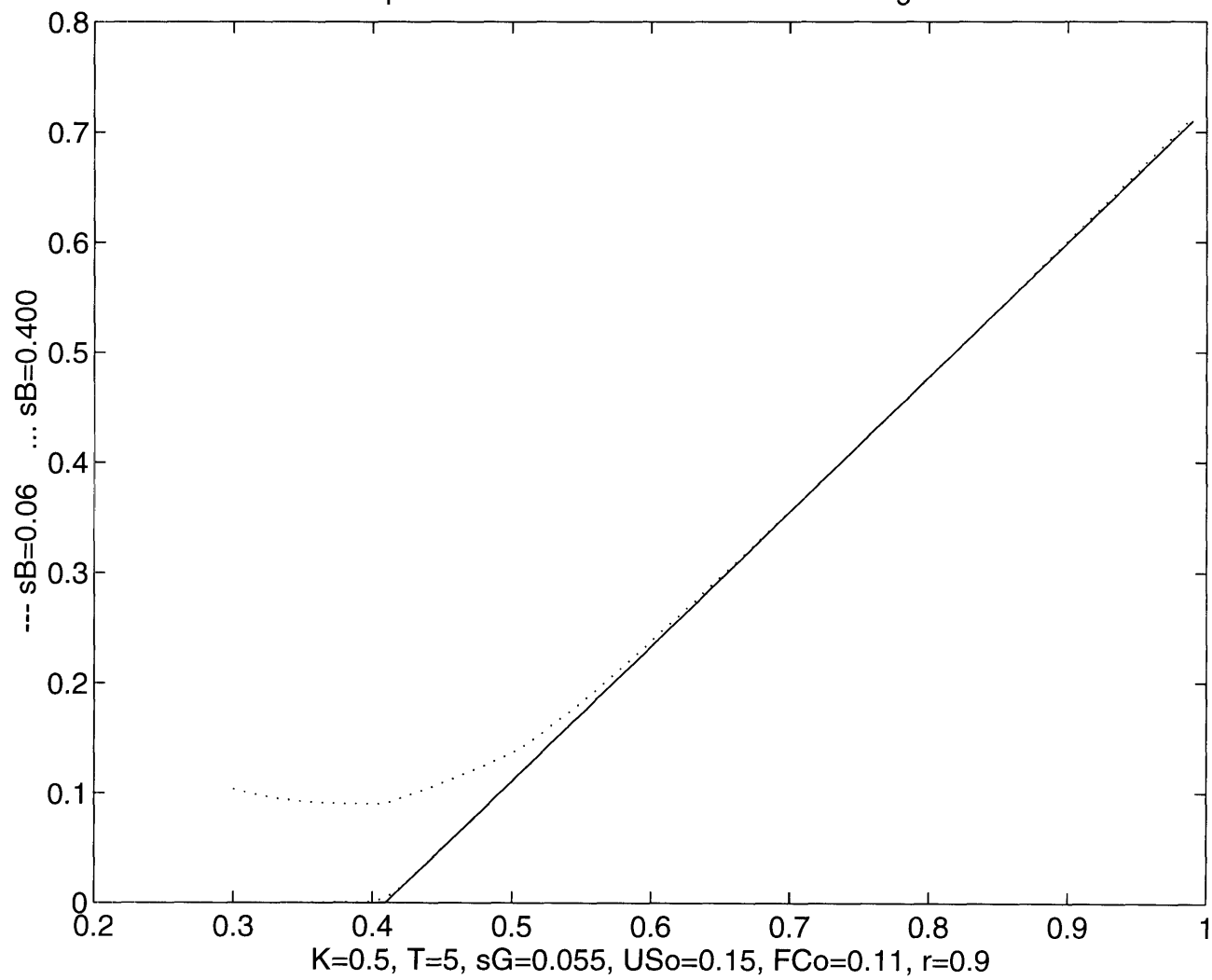




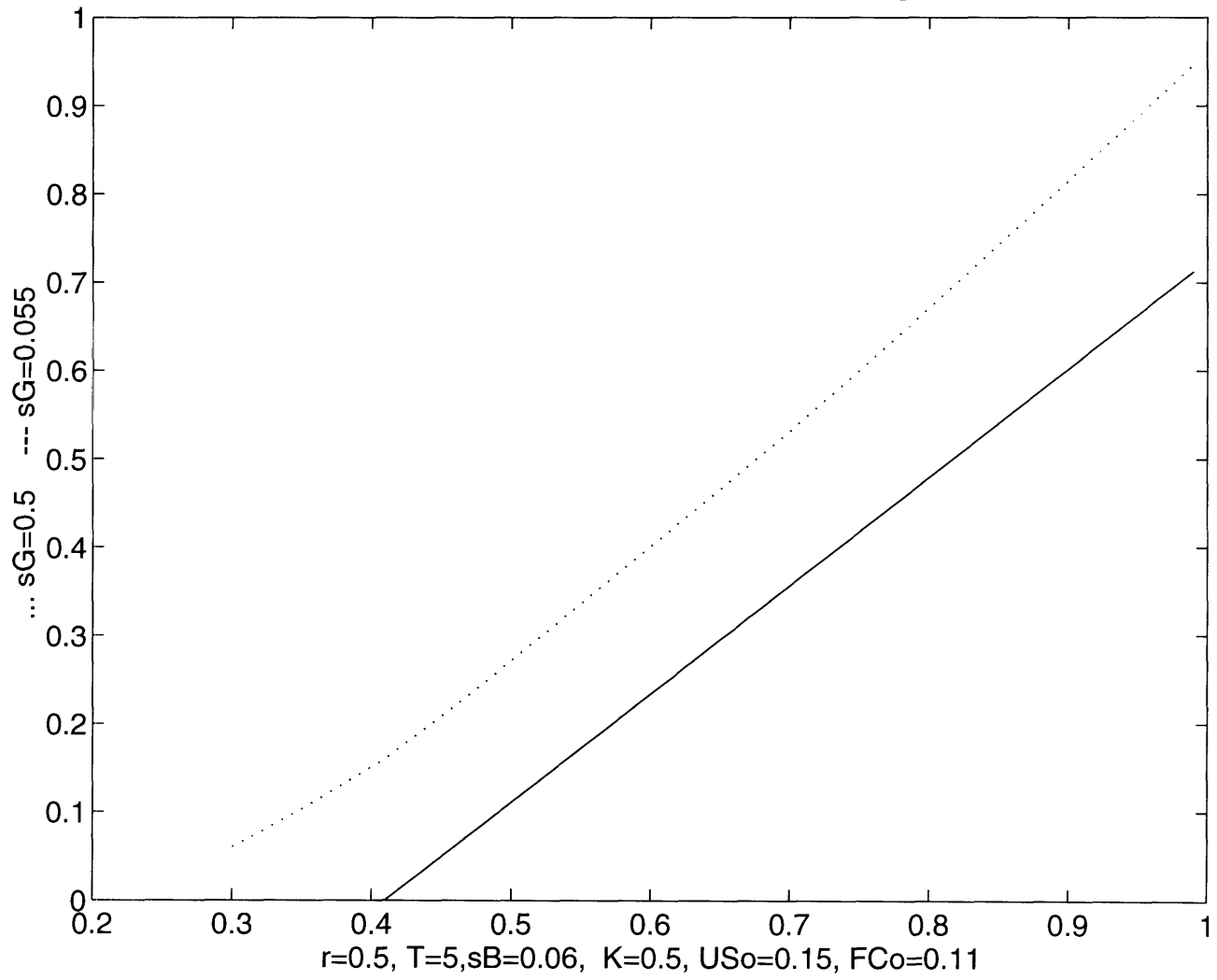
Call Option Price as a function of the Exchange Rate



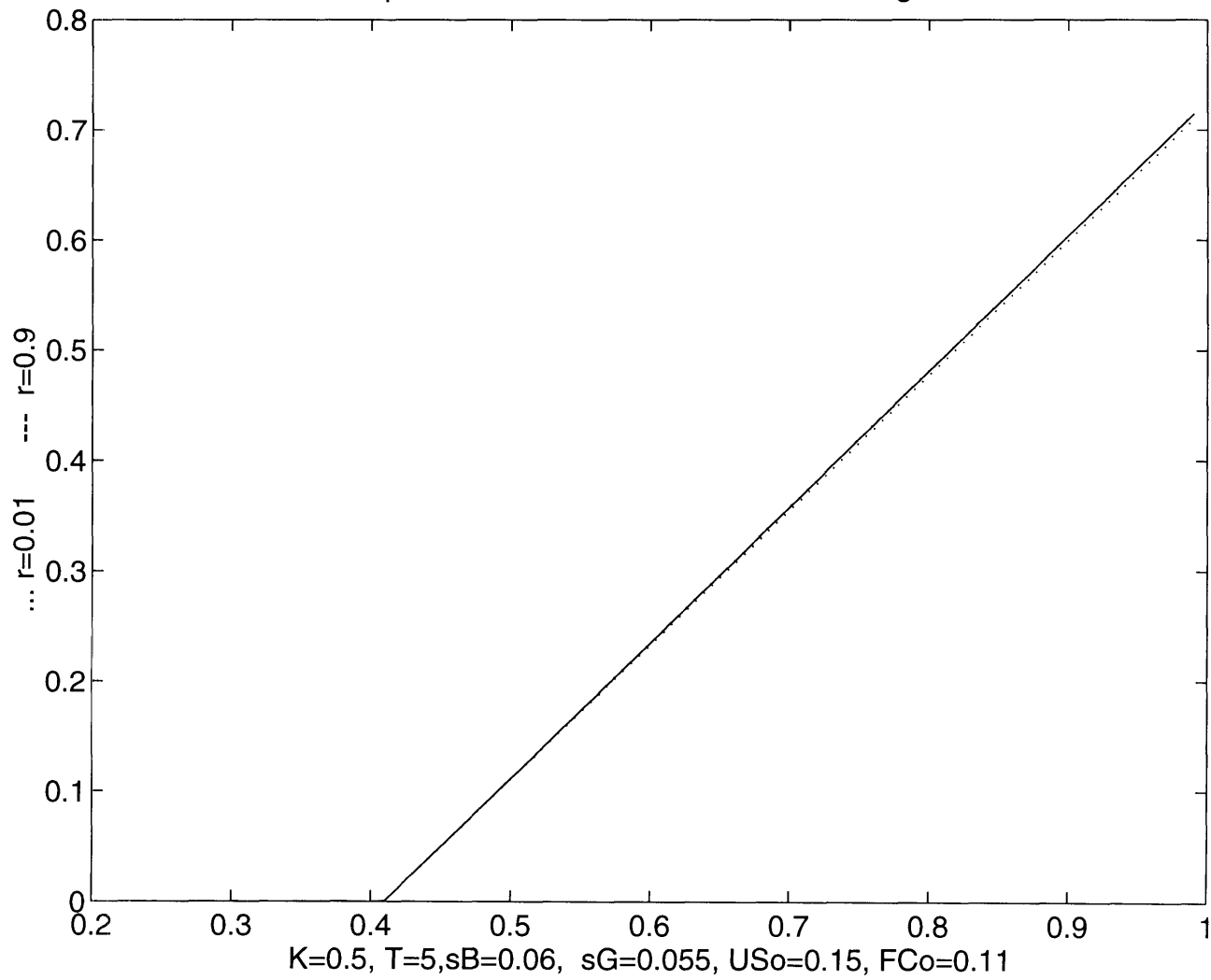
Call Option Price as a function of the Exchange Rate



Call Option Price as a function of the Exchange Rate

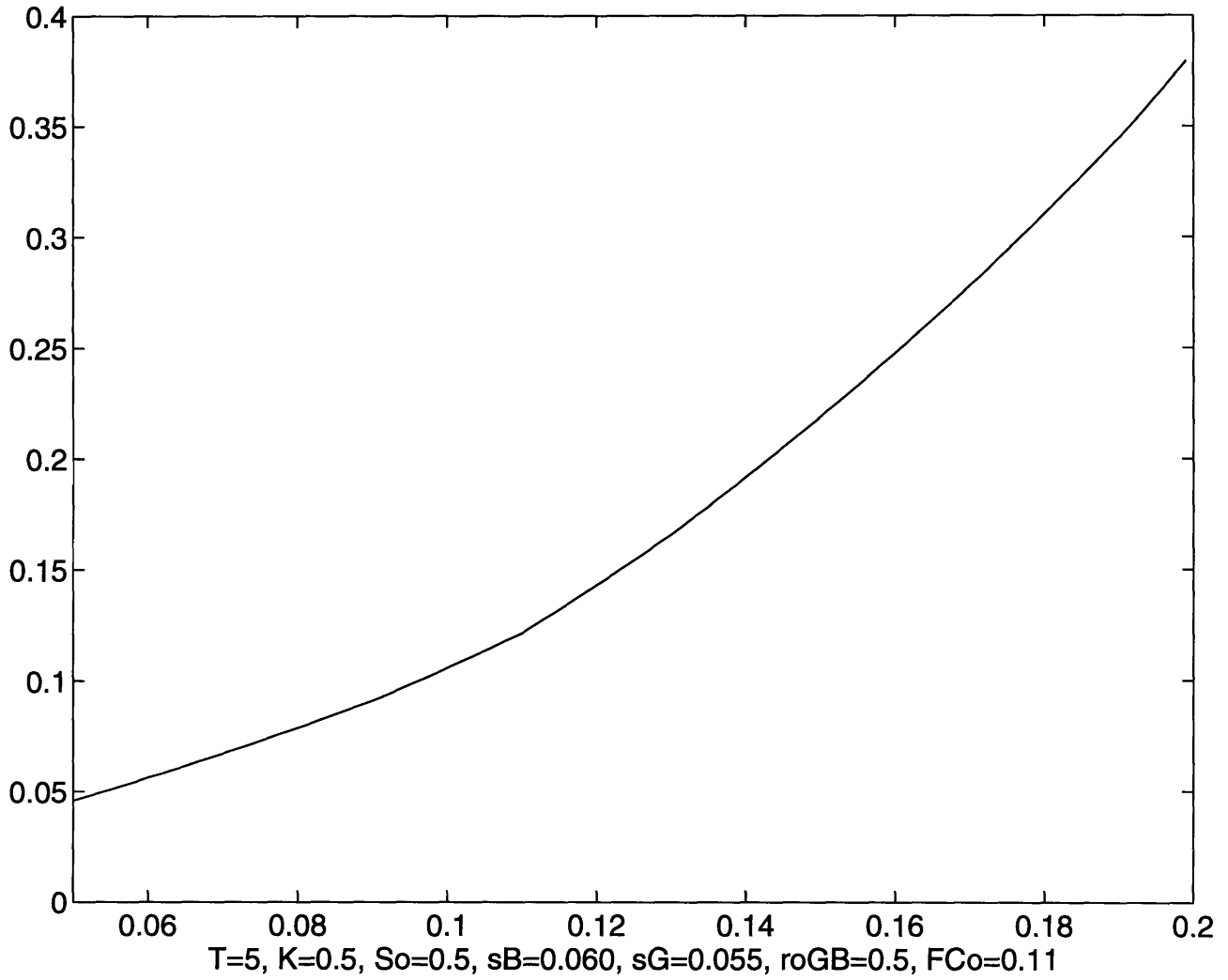


Call Option Price as a function of the Exchange Rate

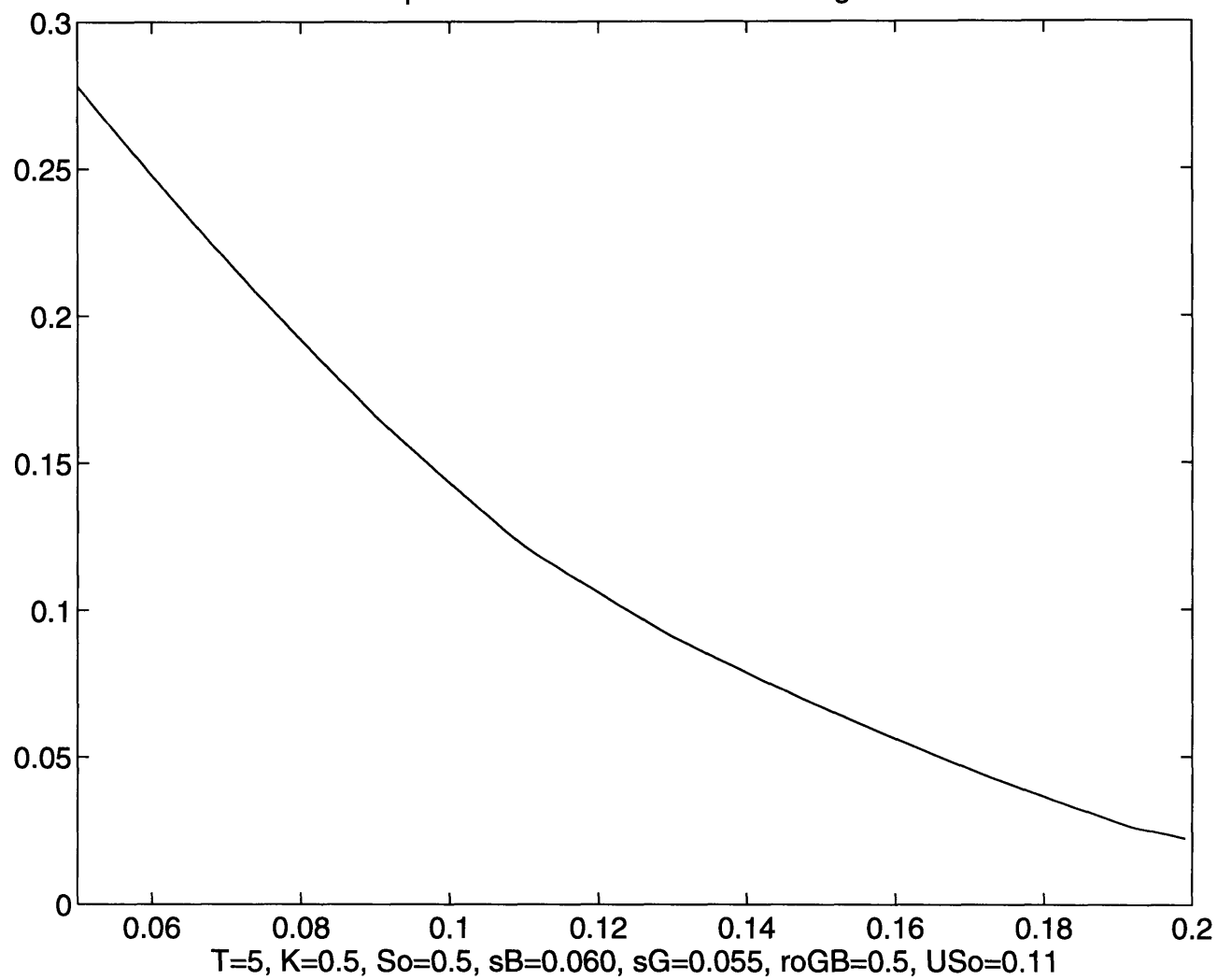


4-16

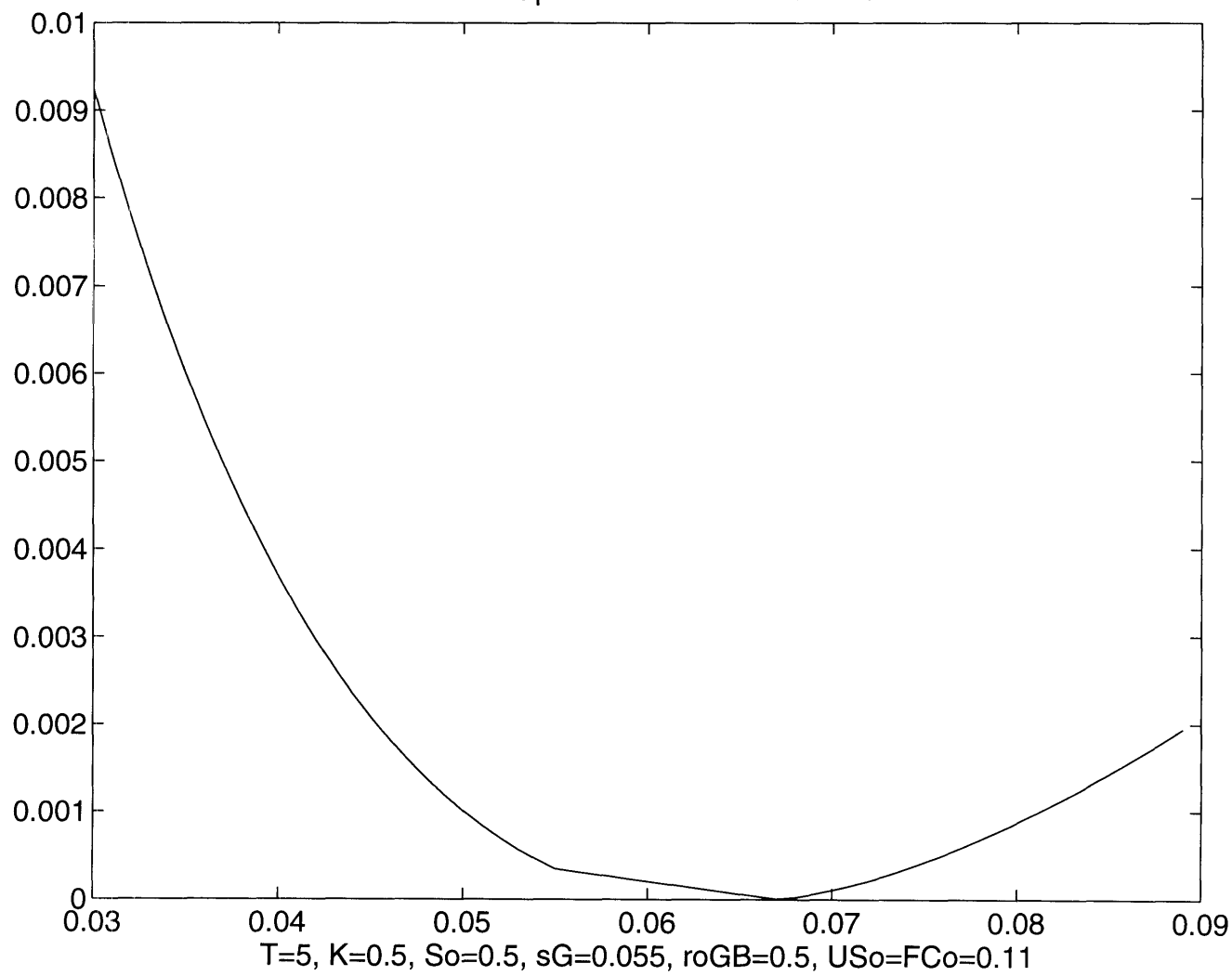
Call Option Price as a function of US interest rate

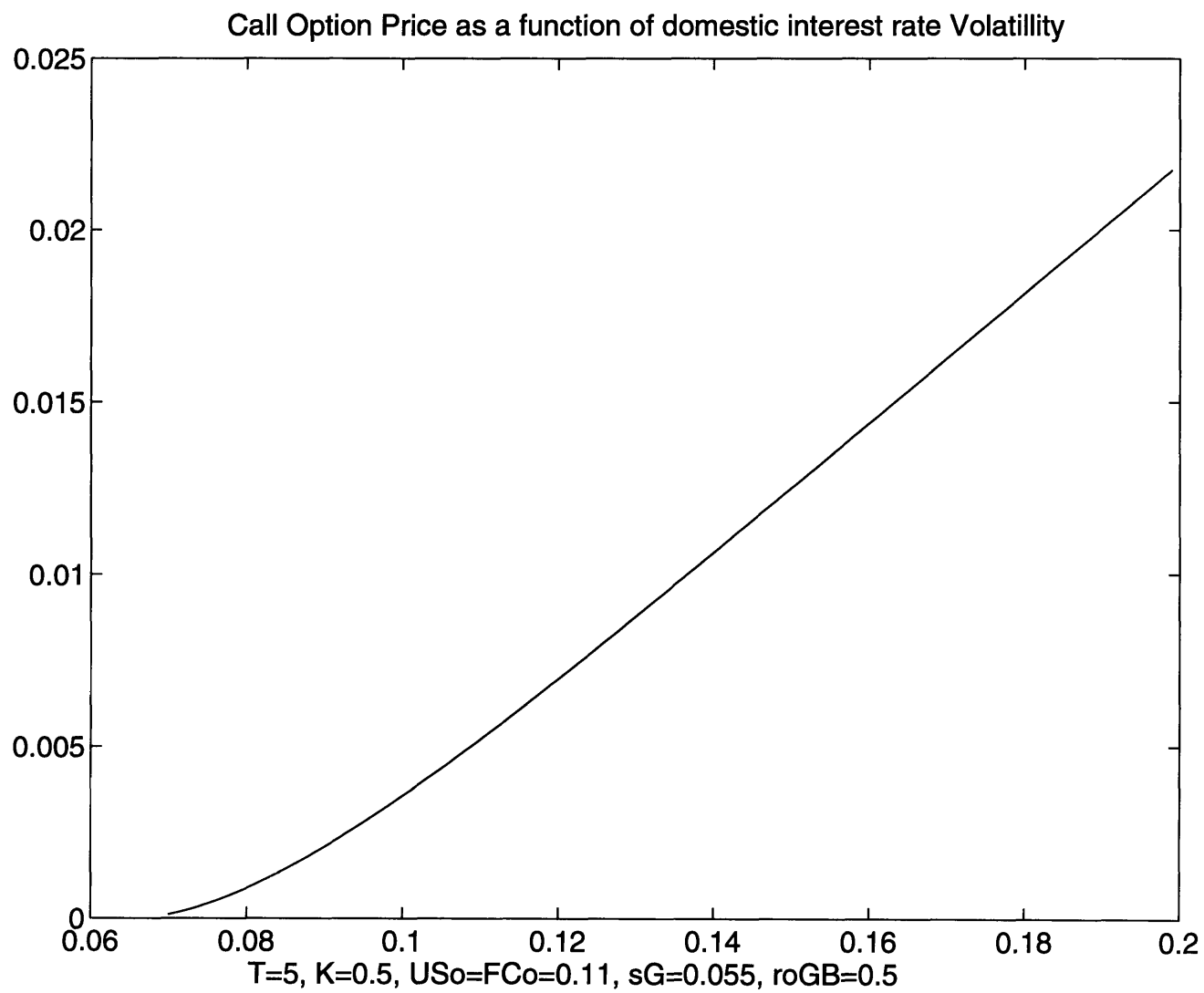


Call Option Price as a function of Foreign interest rate

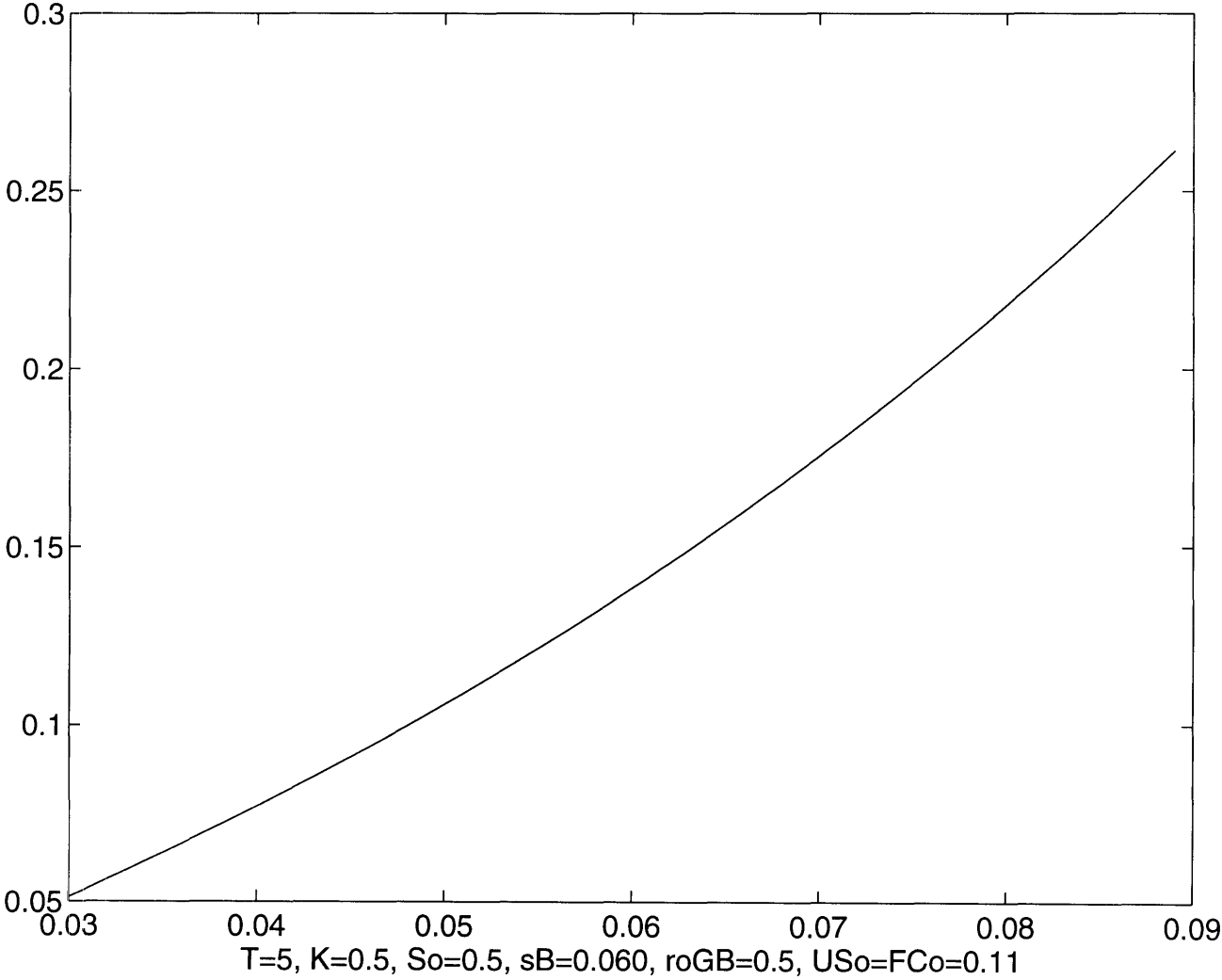


Call Option Price as a function of sB

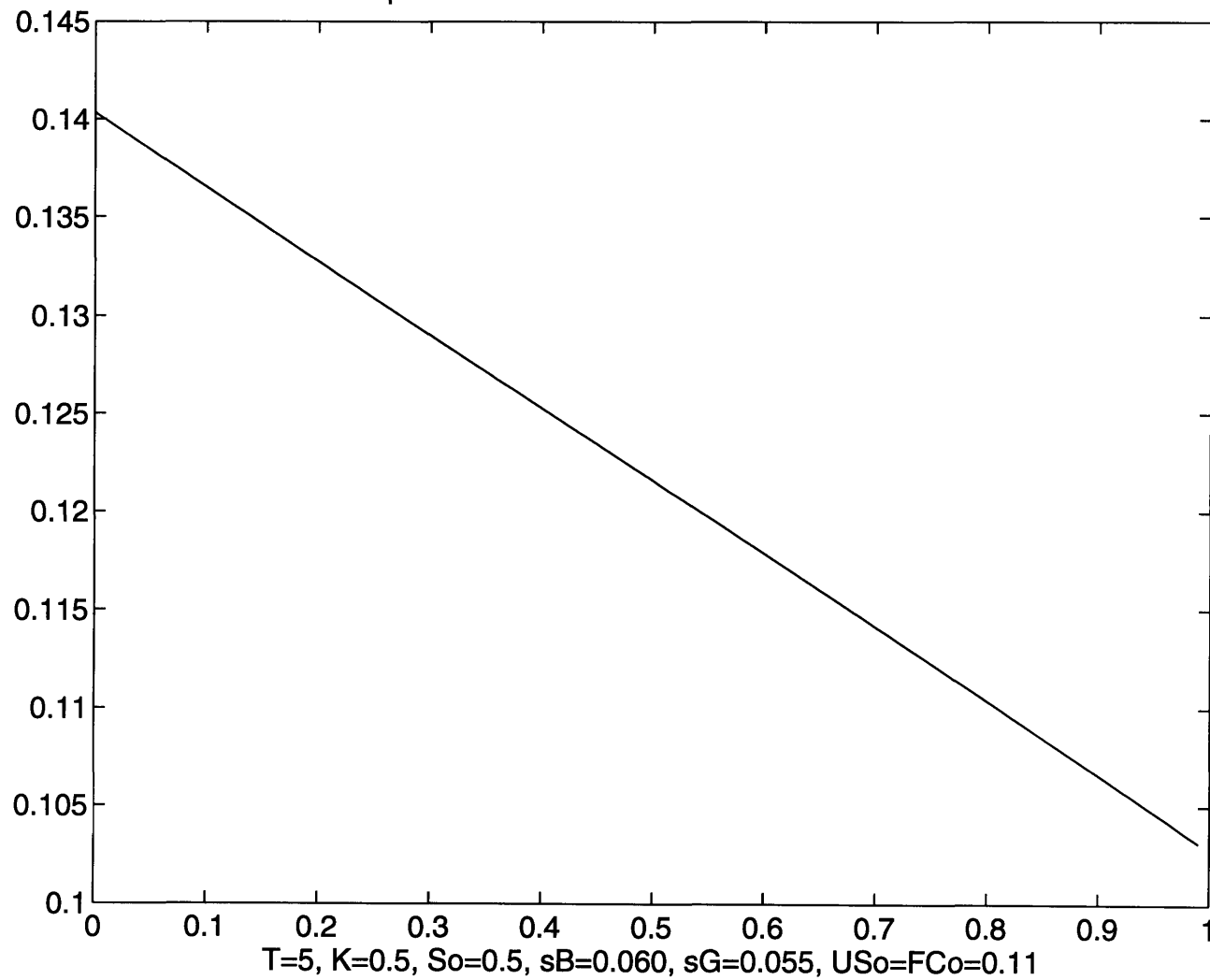


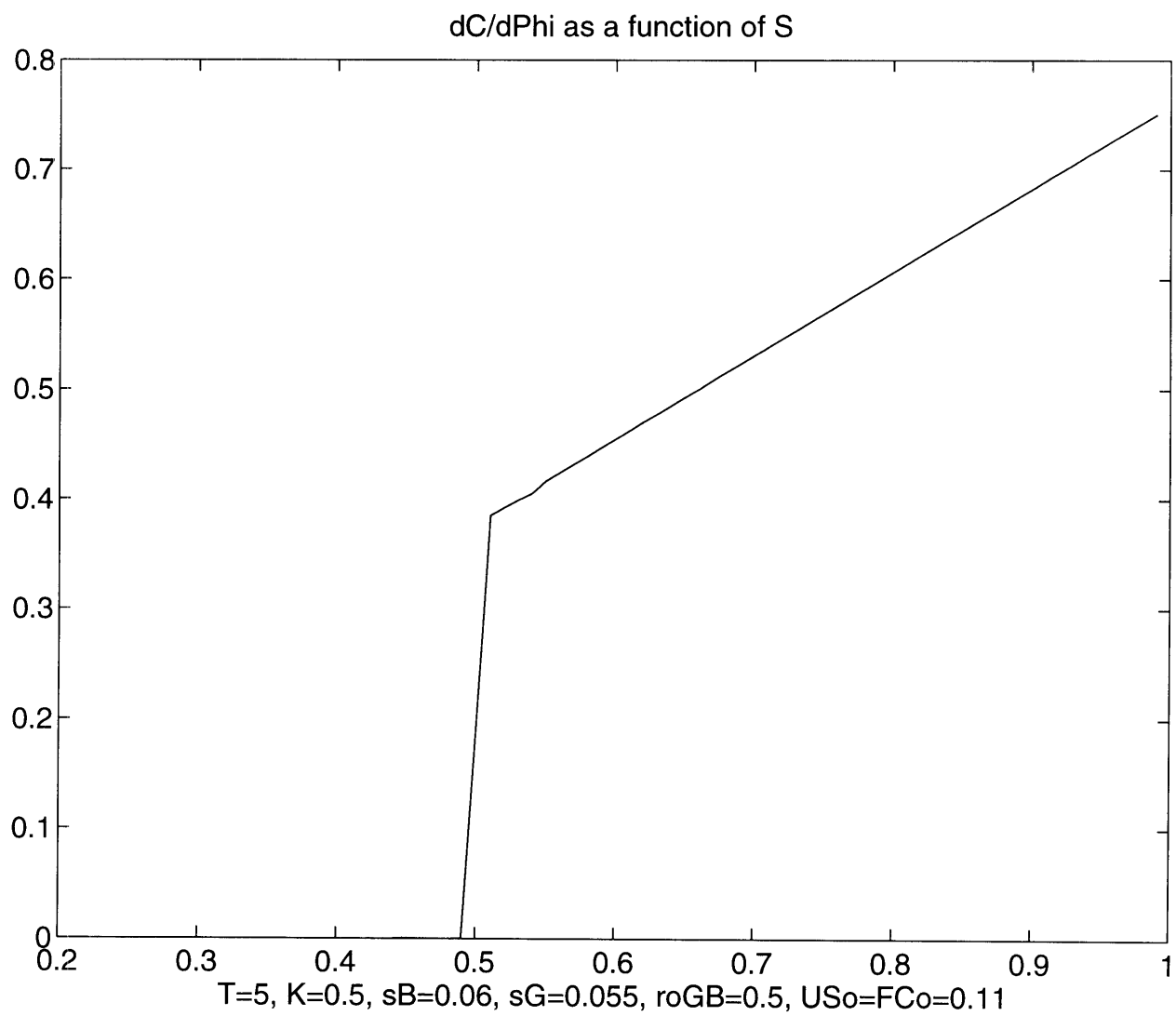


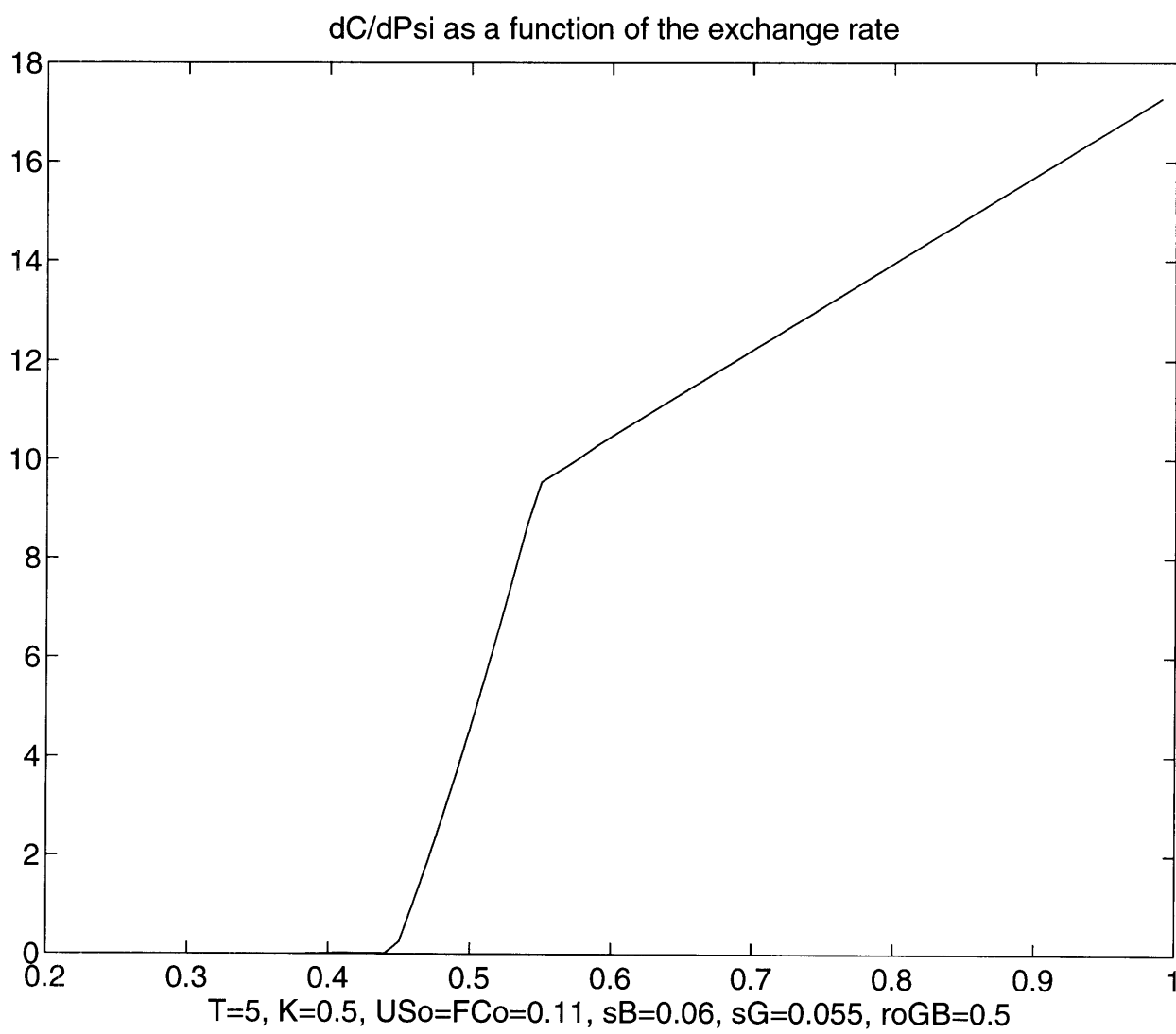
Call Option Price as a function of sG

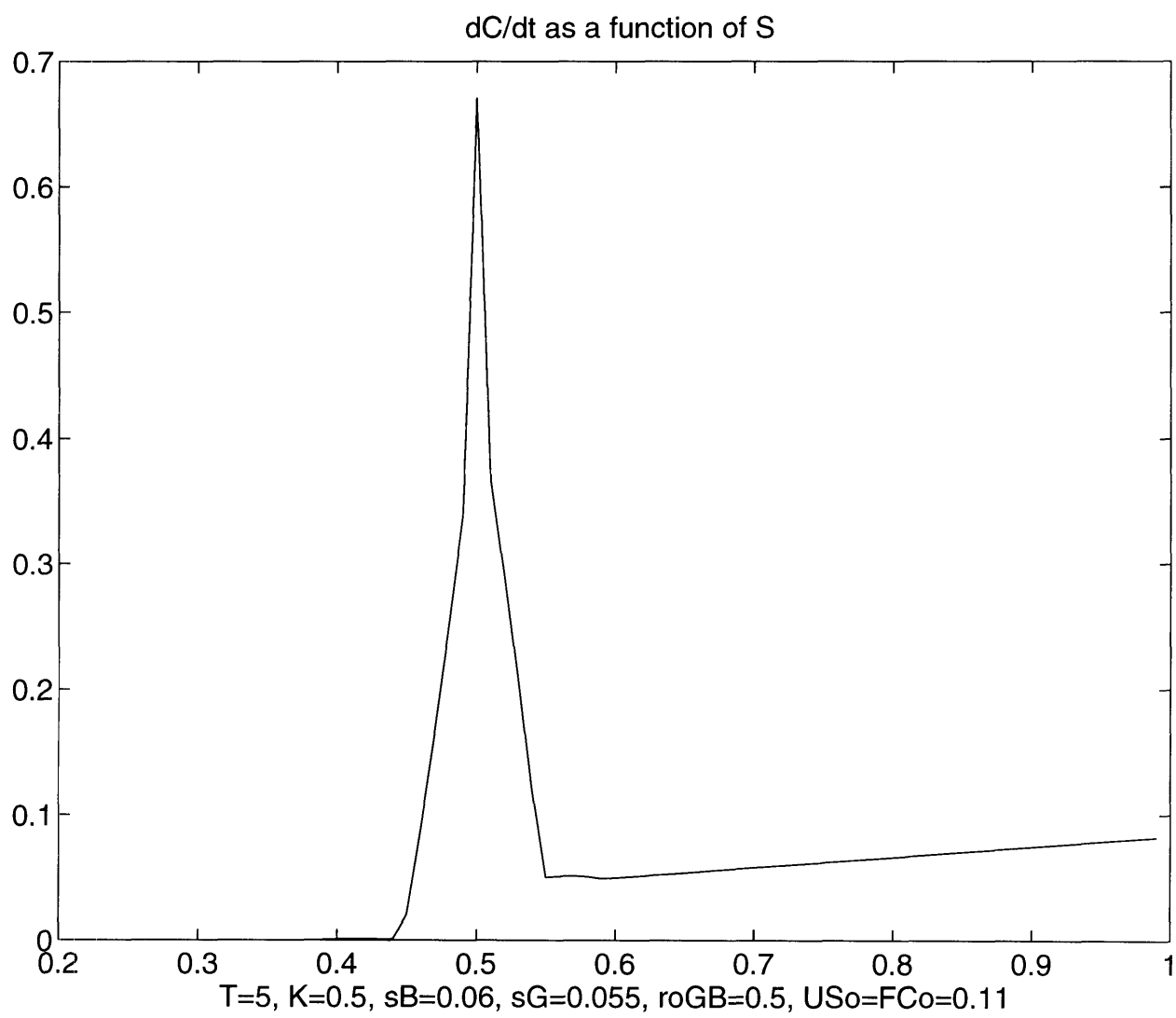


Call Option Price as a function of the Correlation coefficient









47

which is coherent with our previous conclusions. So having the call option price, the current underlying security price, the two interest rates and the striking price, it is possible to deduce the value of the put for an European option with the same underlying security and striking price.

We shall not discuss Dividends, since here the underlying security is not a Stock but an Exchange rate; This means that unlike Exchange Traded Options, these options on Foreign exchange are traded Over The Counter.

An Over the Counter option consists in an agreement between two companies, so there is no intermediate as is the case for the Exchange Traded Market with the Options Clearing Corporation. This absence of intermediate increases the risk considerably but also provides much more freedom to the traders. Indeed in the Over the Counter market there is no standardization on the expiration date: T in our case is the date of the transaction of the company and it will be fixed among the partners.

The same statement stands for the striking price, it could be chosen and fixed by the companies. Consequently there is more possible profit and *excitement* in a Over the Counter exchange and also there are less participants

and competitors in this kind of market comparing to the standard Exchange Traded Market.

This is why this kind of options were used much later than standardised options in the Exchange Traded market. The latter is much more secure and came first and had an educational role for the investors. Once these investors were used to the options markets, they started to use options in other fields such as Foreign Currency.

Now how about a parameter such as the Expected rate of growth of the underlying security? As for the standard options, this parameter characterizes the connection of S and the rest of the market, and again, as in the Exchange Traded Options case, this parameter could not have an increasing influence on the call and a decreasing influence on the put, since we know according to the Put Call parity relation that, other parameters fixed, an increase in c will cause an increase in p as well. So even if it is counter intuitive, the expected rate of growth of S has no influence whatsoever on the value of c .

Now let us see what advantages options on foreign currency offer comparing to a simple portfolio constructed by purchase of Dollars and the Foreign Money.

There are of course the classical advantages offered by options in general, namely the possibility of taking advantage of more attractive transaction costs, margins and taxes. But there is also the important fact that a call option on foreign currency offers a hedge against sudden jumps in the value of the exchange rate. This protection could not be achieved through a stop-loss ordering on S . (*See further*)

But once again there is a further complication in our case, due to the fact that we have two varying interest rates. Indeed if the foreign interest rate is more advantageous than the domestic interest rate, and if we keep the striking price in Dollars, we shall lose money! So it would be more advantageous to exercise the call immediately and have K in foreign currency and take advantage from the more favorable interest rate.

To this point we described only European call options, *ie* options that we can exercise on the expiration day only. But here, we realize that an American call option could be more advantageous, and unlike the European call option, the more time to expiration is long, the more the option is valuable.

There is still more: we saw that the correlation between the two interest rates had a decreasing influence on the value of the call option. Consequently in the case of two independent interest rates the option has a higher value. But if the two interest rates are closely related, which is the case among two partners, the call option is less valuable. However we saw that this influence due to the correlation is minor comparing to interest rates themselves.

Another question that we could ask is whether it is possible to *duplicate* the value of the call option using only the exchange rate and domestic AND foreign bonds. A very similar question has been studied in Cox-Rubinstein [1985] about options on stocks.

The answer is therefore available for a stock: namely we have to buy less than a stock and finance a part of our buying by borrowing money (*ie* selling

bonds) and then buy more by borrowing more if the stock price goes up and sell stock and lend the incoming money if S falls. On the other hand as the time to expiration approaches zero the slope $\frac{\partial c}{\partial S}$ is closer to one as long as $S > K$ and closer to zero otherwise.

How about our case? Buying a stock makes sense of course but what does it mean to buy an exchange rate? and in which currency should we borrow or lend? how about the influence of the expiration time on our policy?

In order to duplicate a call option on foreign exchange, we shall do the following: we buy an amount of foreign currency inferior to one by financing our purchase partly by borrowing Dollars. Therefore the *Value* of our portfolio increases with the value of the exchange rate or equivalently the value of foreign currency if Dollar is considered as fixed. Now we would like that the *Slope* of our portfolio increases as well with S . To have this property we should buy more foreign currency by borrowig more Dollars as the value of the exchange rate goes up and vice versa.

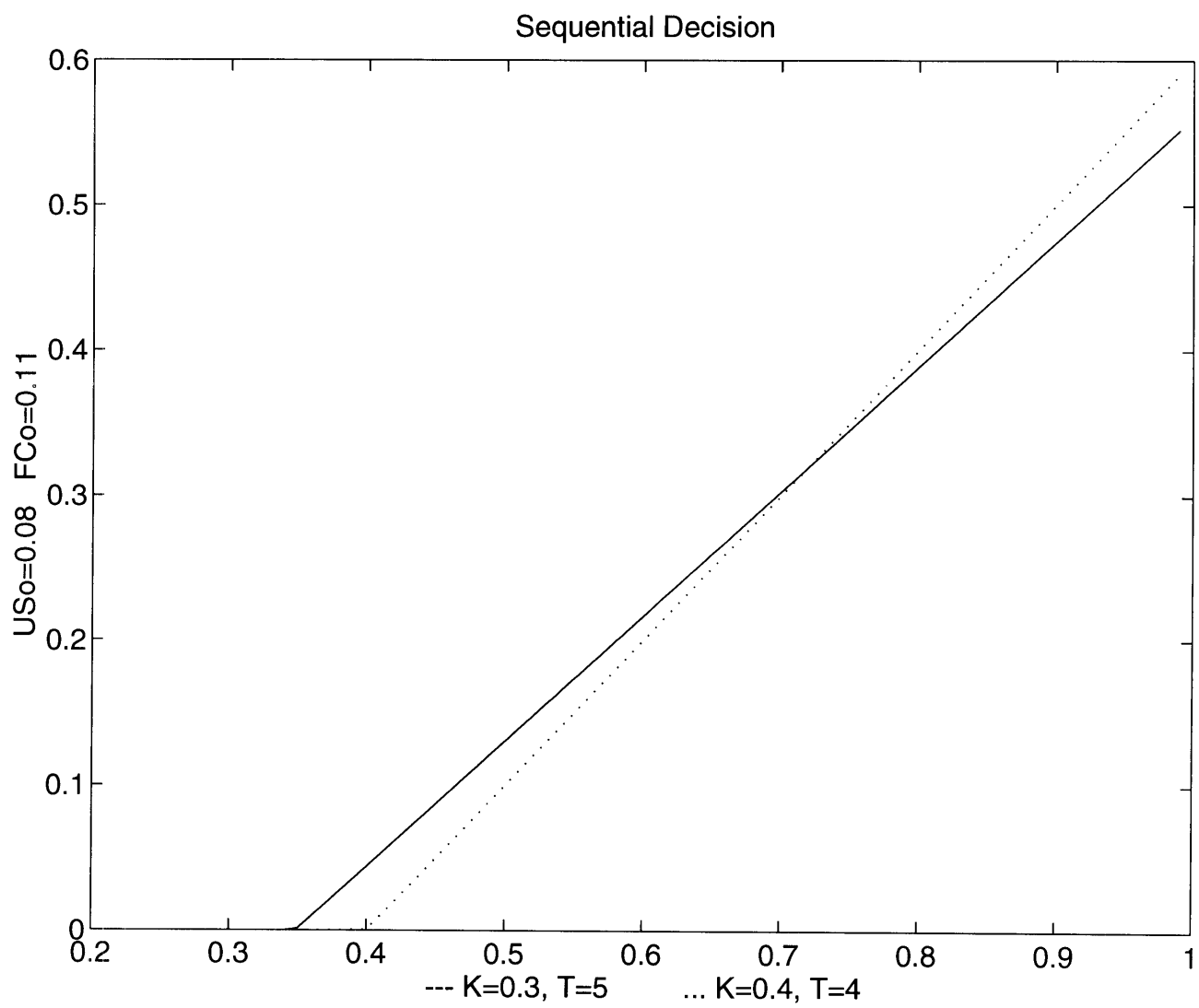
As for the influence of t , time to expiration on our strategy, it totally de-

depends on the comparative values of the domestic and foreign interest rates: if $US_i > FC_i$ then our strategy will be similar to the case of options on stocks otherwise we shall do the contrary.

Of course the same kind of danger (as in the case of stocks) exists in our case, that is the danger of a jump in the exchange rate: if the value of S falls so fast that we can not react to it, we shall lose a lot of money in foreign currency and we will not be able to compensate our borrowing in Dollars. So there is a tracking error: it is not possible to duplicate the value of the call using only foreign currency and bonds. Hence the popularity of options on foreign exchange as we mentioned earlier.

2.6 Sequential Decision System

Now let us remind the fact that in the original paper written by Klaassen-Shapiro, the main problem was to have a *sequential* decision system, which means that we would like to know, for different options with different times to expiration AND different striking prices, which one is more valuable, ie



which one should we choose?

In other words if we have two options, one with a striking price K_1 and the other one with a striking price K_2 ; one with a maturity T_1 and the other one with a maturity T_2 , which one is worth more?

The easiest way to answer to this question would be to plot the two corresponding curves providing the values of c_1 and c_2 for different present exchange rate values, and then to take the present exchange rate value and compare c_1 and c_2 .

Obviously the question becomes interesting only if time to expiration and striking price act in opposite ways. For example we could consider a call option with a maturity 4 and a striking price 0.4 and another one with $T = 5$ and $K = 0.3$, with the further assumption that $US_i < FC_i$.

As we can see on the curves plotted for $0.2 < S < 1$, the two call option functions intersect at S close to 0.72; Consequently if the present value of the exchange rate is inferior to 0.72, the option with the striking price $K = 0.3$ is more valuable, and otherwise it is the call with the higher striking price

which should be preferred.

2.7 An Exact formula

It might seem paradoxical that we should talk about an exact pricing formula; indeed we originally used Grabbe's exact formula to derive our decision tree. But Grabbe's generalized Black-Scholes formula is a Continuous formula and what is more we used his differential equation to construct our tree. So the idea of using this tree in order to have an exact *discrete* formula is a natural idea. It is the same idea that pushed Prof. Sharpe to redemonstrate Black Scholes formula in a much simpler way than the way Black and Scholes did it themselves.

So let us consider the following problem: We have our 4-ary tree with upward and downward transitions for the two parameters Φ and Ψ . What is more, the upward and downward transitions are symmetric so an upward transition followed by a downward transition is equivalent to a horizontal transition.

So we could take the following notation:

$$\Phi(j) = \Phi_0 + j\Delta\Phi$$

$$\Psi(k) = \Psi_0 + k\Delta\Psi$$

where Φ_0 , Ψ_0 , $\Delta\Phi$ and $\Delta\Psi$ are determined as we already explained.

On the other hand with our previous approximations we can write on the expiration day:

$$S(j, k) = \exp\left(\frac{\Phi(j) + \Psi(k)}{2\sigma_B} - \frac{\Phi(j) - \Psi(k)}{2\sigma_G}\right)$$

and finally as we know the price of the call option on the expiration day is:

$$c_f(j, k) = \max(0, S(j, k) - K)$$

Now how could we have the present value of the call price as a function of these $c_f(j, k)$'s? To answer to this question we could observe the binary

tree. Indeed what we have here is the cartesian product of two binary trees. In the binary case we use a Newton formula $(pa + qb)^n$ where a represents an upward transition, b a downward transition, n the number of stages, p the probability of an upward transition and q the probability of a downward transition, with $p + q = 1$.

If, as it is our case, the upward and downward transitions are symmetric we can write $b = \frac{1}{a}$. So the present value of the call, in the binary case will be:

$$c = (pa + \frac{q}{a})^n = \sum_{k=0}^n C_n^k p^k q^{n-k} a^{2k-n}$$

where C_n^k is the binomial coefficient $\frac{n!}{(n-k)!k!}$

Now our case is very similar to this one, save for the number of possible transactions. We could call a the event of an upward transaction for Φ or equivalently j , and therefore $\frac{1}{a}$ the symmetric downward transition. In the same way b and $\frac{1}{b}$ will be the upward and downward transitions for Ψ or equivalently k . The corresponding probability of the two first events is p and

the probability of the two last events is q , so that $2p + 2q = 1$. With these notations we shall have:

$$c = (pa + \frac{p}{a} + qb + \frac{q}{b})^n$$

if we call r_1, r_2, r_3 and r_4 for integers between 0 and n we have the following

Newton expansion:

$$c = \sum_{r_1+r_2+r_3+r_4=n} \frac{n!}{r_1!r_2!r_3!r_4!} p^{r_1+r_2} q^{r_3+r_4} c_f(r_1 - r_2, r_3 - r_4)$$

This is our *discrete exact pricing formula* . Note that this equation provides a link between the Newton's coefficients and j and k since:

$$j = r_1 - r_2$$

and

$$k = r_3 - r_4$$

Logically, if we take our discrete relation and use a passage to limit, we shall find the same continous generalized Black Scholes formula as the one in the

first section.

2.8 Implied Volatility

One of the most critical issues in the determination of the prices of call options is the stock volatility. The first idea we had while writing this thesis was to take a reasonable value for the volatility (and the correlation coefficient!) and then to compute the value of the call option and compare this value with the real value available in the market and see if there was any opportunity to make money.

But once again, this will not solve the main problem of volatility. So the subsequent idea is to solve the reverse problem. In other words we could take the real value of the call option and plug it in the model and then we could *deduce* the value of our volatility. But then again, we have TWO volatilities σ_B and σ_G plus a correlation coefficient ρ_{GB} . However it is reasonable to choose a fixed value for σ_B and ρ_{GB} , since the variations of the interest rates are of secondary importance comparing to the variations of the exchange rate itself.

The parameter to be evaluated is therefore σ_G with $G = SB^*$. The method to be used is Dichotomy, since the “function” $c(\sigma_G)$ has a very complicated expression and is hard to invert. In other words, since we know that c increases with σ_G , after having taken c_{real} the real market value of the call option, if for a given σ_G , we have $c > c_{real}$ we shall decrease the value of the volatility and vice versa.

But upon reflection, this method is not necessary! Indeed a *Graphic* method will be enough for our purpose. We could take the curve of c as a function of σ_G , other parameters fixed and then take c_{real} on the vertical axis and deduce directly the implied value of σ_G .

3 Conclusion

Two main ideas are to be found in this thesis: First, the generalization of the Black-Scholes formula in the case of varying interest rates as suggested by Grabbe and then, the discretization of this formula and its implementation on computer. Once the computational experience was carried out, we checked

the consistency of the corresponding results. Note that the discretization was necessary to form a decision tree allowing an evaluation of different strategies.

We started this thesis with the idea of generalizing the Black Scholes formula on Foreign Currency to the case of stochastic interest rates. But at the end of the thesis we saw that this method could be used in another way: a tool to evaluate different parameters (stochastic or financial) *given* the real call price. In particular, we can compute the implied volatility.

Consequently, further research could be carried out in this field, specially for the evaluation and the *Identification* of the volatility using “suitable” data.

References :

Cox, J.C. and M. Rubinstein (1985), OPTIONS MARKETS, *Prentice-Hall*.

Ho, T.S.Y and S.B. Lee (1986), TERM STRUCTURE MOVEMENTS AND PRICING INTEREST RATE CONTINGENT CLAIMS. *The Journal of Finance*, VOL. XLI, NO. 5.

Klaassen Pieter, J.F. Shapiro, D.E. Spitz (1990), SEQUENTIAL DECISION MODELS FOR SELECTING CURRENCY OPTIONS. *International Finance Services Research Center No.133-90, Massachusetts Institute of Technology*.

Grabbe J.O. (1983), THE PRICING OF CALL AND PUT OPTIONS ON FOREIGN EXCHANGE. *Journal of International Money and Finance*, 2, 239-253.

Hull J. and A. White (1990), VALUING DERIVATIVE SECURITIES USING THE EXPLICIT FINITE DIFFERENCE METHOD. *Journal of Financial and Quantitative Analysis*, VOL 25, NO. 1.

Heath David, R. Jarrow and A. Morton (1987), BOND PRICING AND THE TERM STRUCTURE OF INTEREST RATES : A NEW METHODOLOGY. *School of Management. Cornell University*.

Hull J. and A. White (1992), ONE FACTOR INTEREST RATE MODELS AND THE VALUATION OF INTEREST RATE DERIVATIVE SECURITIES. *Faculty of Management, University of Toronto*.

Garman M.B. and S.W. Kohlhagen (1983), FOREIGN CURRENCY OPTION VALUES. *Journal of International Money and Finance*, 2, 231-237.

Biger N. and J. Hull (1990), THE VALUATION OF CURRENCY OPTIONS. *Faculty of Administrative Studies, York University*.

Biger N. and Hull (1983), THE VALIDATION OF CURRENCY OPTIONS, *Financial Management* 24-28.

Oksendal B. (1984), STOCHASTIC DIFFERENTIAL EQUATIONS. *Springer and Verlag*.

Kelly A. and I. Pohl (1984), A BOOK ON C. *Benjamin/Cummings Publishing Company, Inc.*

Papoulis A. (1986), PROBABILITY, RANDOM VARIABLES AND STOCHASTIC PROCESSES. *Mc Graw Hill*.

Geske R. and K. Shastri (1985) VALUATION OF APPROXIMATION: A COMPARAISON OF ALTERNATIVE APPROACHES *Journal of Financial and Quantitative Analysis*, 20.

Klaassen Pieter (1993), ALMnewval.c *Operations Research and Management Department, Massachusetts Institute of Technology*.